Harmonic analysis of M.C's & RW's

$Pf(x) = \sum_{y \in S} P(x,y) f(y)$

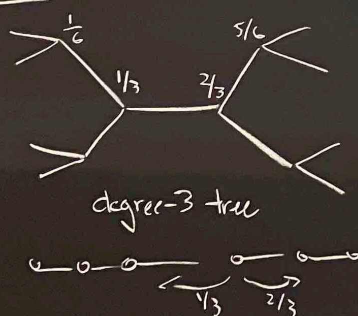
P = tr. kernel on S, $h: S \to \mathbb{R}$ is **P-harmonic** if $P|h| < \infty \wedge Ph = h$

Showed: • MC irreducible & recurrent: h-P-harmonic + $h \geq 0$ or $h \in \ell^\infty \Rightarrow h = \text{constant}$.

• SRW on $\mathbb{Z}^d$, $d \geq 1$, has no non-constant bounded harmonic function ← Liouville.

Maybe one can use ≥ 0 harmonic functions?

Ex M.C. w/o Liouville property



projection onto $\mathbb{Z}$

$$h(x) = P^x\left(\lim_{n\to\infty} \overline{X_n} = +\infty\right)$$

$$h(n) = \begin{cases} 1 - \frac{1}{3} 2^{-n+1} & n \geq 0 \\ \frac{1}{3} 2^n & n \leq 0 \end{cases}$$ is harmonic.

Notice: $\mathcal{E} := \{ X^{-1}(B) : B \in \Sigma^{\otimes \mathbb{N}} \wedge \theta^{-1}(B) = B \}$ ← shift invariant events.

shift invariant events

Lemma $\forall A \in \mathcal{E}$: $h(x) := P^x(X \in A)$ is P-harmonic.

Pf Definition ensures $\{X \in A\} \underset{A = X^{-1}(B)}{=} \{(X_0, X_1, \ldots) \in B\} \overset{B = \theta^{-1}(B)}{=} \{(X_1, X_2, \ldots) \in B\}$

So $P^x(X \in A) = P^x((X_1, X_2, \ldots) \in B) = \sum_{y \in S} P(x,y) P^y((X_0, X_1, \ldots) \in B) = \sum_{y \in S} P(x,y) h(y)$

Thm Suppose MC irreducible. Then $\forall x, y \in S$: $P^x|_{\mathcal{E}} \ll P^y|_{\mathcal{E}}$.

Moreover, for all P-harmonic $h \in \ell^\infty(S)$:

$$Z_h := \lim_{n \to \infty} h(X_n) \quad \text{exists } P^x\text{-a.s. } \forall x \in S.$$

and the map $h \mapsto Z_h \mathbb{1}_{\{\lim_{n\to\infty} h(X_n) \text{ exists}\}}$ is a bijection

$$\{h \in \ell^\infty(S) : Ph = h\} \longrightarrow \{Z \in L^\infty : \mathcal{E}\text{-meas}\}$$ with inverse $h(x) = E^x(Z)$.

Pf: $A \in \mathcal{E}$. Then $P^x(X \in A) = P^x(\{(X_n, X_{n+1}, \dots) \in X^{-1}(A)\}$

$$\geq P^n(x,y)\, P^y(X \in A).$$

So $P^y(X \in A) \leq \frac{1}{\underbrace{\sup_{n \geq 0} P^n(x,y)}_{>0 \text{ by irreducibility}}} P^x(X \in A)$

Now $h \in \ell^\infty$, $Ph = h$

$\Rightarrow h(X_n)$ is bounded martingale under P^x, $\forall x \in S$.

So $Z_h := \lim_{n \to \infty} h(X_n)$ exists P^x-a.s. $\forall x \in S$.
& L^1

Also $h(x) = E^x(h(X_n)) \xrightarrow[n \to \infty]{} E^x(Z_h)$.

Conversely, by argument in previous lemma:

$Z \in L^\infty$, $\mathcal{E}$-meas $\Rightarrow h(x) = E^x(Z)$ is P-harmonic bdd.

This is the bijection.

Now Random Walks:

<u>Thm</u> (Choquet - Deny) Let $Q: \mathbb{Z}^d \to [0,1]$ be s.t. $\sum_{x \in \mathbb{Z}^d} Q(x) = 1$.
Assume $P(x,y) := Q(y-x)$ is irreducible. Then $\forall h: \mathbb{Z}^d \to \mathbb{R}_+$:
h is P-harmonic $\iff \exists! \mu_h \in \mathcal{M}_1(\mathbb{R}^d)$ s.t.

(1) $\mu_h\left(\left\{t \in \mathbb{R}^d : \sum_{x \in \mathbb{Z}^d} Q(x) e^{t \cdot x} = 1\right\}\right) = 1$ exponentials that are P-harmonic

(2) $\forall x \in S: \quad h(x) = h(0) \int \mu_h(dt) \exp\{t \cdot x\}$.

$$\forall x \in S: \quad \sup_{n \geq 0} P^n(x,0) \leq h(x) \leq \frac{1}{\sup_{n \geq 0} P^n(0,x)}$$

<u>Pf</u> $1 = h(0) = \sum_{y \in S} P^n(0,y) h(y) \geq P^n(0,x) h(x)$, $\quad h(x) = \sum_{y \in S} P^n(x,y) h(y) \geq P^n(x,0) h(0) = P^n(x,0)$

<u>Note</u> $\mathcal{C} := \left\{ h \in \mathbb{R}_+^S : Ph \leq h \wedge h(0) = 1 \right\}$ ... compact subset of $\mathbb{R}^S$.

Lemma $h : S \to \mathbb{R}_+$, P-harmonic, $h>0$. Then

$$P_h(x,y) = P(x,y)\,\frac{h(y)}{h(x)}$$

is transition kernel.

h-transformed Markov chain

Pf $\displaystyle\sum_y P_h(x,y) = \sum_y P(x,y)\,\frac{h(y)}{h(x)} = \frac{h(x)}{h(x)} = 1$ $\boxed{\times}$

Pf of Thm: $Ph = h$, $h(0)=1 \Rightarrow h_z(x) := \dfrac{h(x+z)}{h(z)}$ also obey $Ph_z = h_z$, $h_z(0)=1$.

Now
$$h_y(x) = \frac{1}{h(y)}\,h(x+y) = \frac{1}{h(y)}\sum_{z\in\mathbb{Z}^d} P(x+y,z)\,h(z)$$
$$= \frac{1}{h(y)}\sum_{z\in\mathbb{Z}^d} \underbrace{P(x+y,x+z)}_{=P(y,z)}\,h(x+z) = \frac{1}{h(y)}\sum_{z\in\mathbb{Z}^d} P(y,z)\,h(z)\,h_z(x) = \sum_{z\in\mathbb{Z}^d} P_h(y,z)\,h_z(x)$$

We showed: $\forall x\in\mathbb{Z}^d$: $\{h_{X_n}(x)\}_{n\ge 0}$ is martingale under P_h^0 = law of h-transformed M.C.

claim $\forall x \in \mathbb{Z}^d$; $\bar{h}(x) := \lim_{n\to\infty} h_{X_n}(x)$ exists P_0^y-a.s. $\forall y \in S$.

Pf Martingale converge + boundedness.

claim $\forall x \in S$: $h(x) = E_h^0(\bar{h}(x))$ (*) (by boundedness)

claim $P_h^0(\bar{h}(x) \neq P\bar{h}(x)) = 0$ $\forall x \in S$.

Pf h_x harmonic so $h_{X_n}(x) = \sum_{y\in S} P(x,y)\, h_{X_n}(y)$ ... by Fatou: $\bar{h} \geq P\bar{h}$.

But $h(x) = E_h^0(\bar{h}(x)) \geq E^0(P\bar{h}(x)) = Ph(x) = h(x)$

claim $\forall x,y \in \mathbb{Z}^d$: $\bar{h}(x+y) = \bar{h}(x)\,\bar{h}(y)$ P-a.s. $\forall z \in S$.

Pf $h_z(x+y) = \dfrac{h(x+y+z)}{h(z)} = \dfrac{h(x+z)}{h(z)} \dfrac{h(x+z+y)}{h(x+z)} = h_z(x)\, h_{x+z}(y)$ $\boxtimes$

Fact: irreducibility $\Rightarrow$ $h_{z+X_n}(x) \longrightarrow \bar{h}(x)$ $\forall z \in \mathbb{Z}^d$ a.s. $\bar{h}$ harmonic $\Rightarrow \sum_x Q(x) e^{t\cdot x} = 1$

Define $t_j := \log \bar{h}(e_j)$ $j = 1 \dots d$. Then $\bar{h}(x) = \exp\{t\cdot x\}$. ...

Let μ_h be law of $(\log \bar{h}(e_1), \dots, \log \bar{h}(e_d))$ on $\mathbb{R}^d$. Then (*) becomes $h(x) = \int \mu_h(dt) \exp\{t\cdot x\}$.