Ergodic Theory

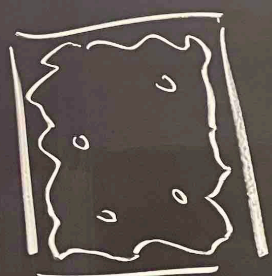
$\partial\Lambda_n = \{(x,y)\in\mathbb{Z}^d : x\in\Lambda_n,\ y\notin\Lambda_n,\ \|x-y\|_1 = 1\}$

Thm (Spatial Ergodic Thm)

Let $(\mathcal{X},\mathcal{G},\mu)$ = meas. space, $\{\varphi_x : x\in\mathbb{N}^d\}$ m.p.t's s.t. $\forall x,y\in\mathbb{N}^d$: $\varphi_{x+y}=\varphi_x\circ\varphi_y$. Then

$\forall f\in L^1\ \exists \bar{f}\in L^1$ s.t. $\forall\ \Lambda_n\subseteq\mathbb{N}^d$ s.t.

- $\Lambda_n \subseteq [0,n)^d\cap\mathbb{N}^d =: B_n(0)$
- $\liminf \frac{|\Lambda_n|}{n^d} > 0 \ \wedge\ \lim_{n\to\infty}\frac{|\partial\Lambda_n|}{n^d}=0$



Then $\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n} f\circ\varphi_x \longrightarrow \bar{f}$ a.e.

If $\mu(\mathcal{X})<\infty$, then convergence in L^1.

Pf

Λ_n

$\Lambda_n, \|x-y\|_1 = 1\}$

$\}$ m.p.t's.

s.t.

0

Lemma (Max ineq.) $f^* := \sup_{n \geq 1} \frac{1}{|B_n(0)|} \sum_{x \in B_n(0)} |f| \circ \varphi_x$

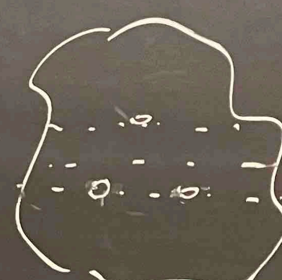
Then $\forall \lambda > 0 \quad \mu(f^* > \lambda) \leq \frac{6^d}{\lambda} \|f\|_1$

Pf of Thm: 1) Convergence checked directly for f of form $f = g + \sum_{i=1}^{d} (h_i - h_i \circ \underbrace{\varphi_i}_{=\varphi_{e_i}})$

$g = g \circ \varphi_i \quad i = 1..d$

NTS $\frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} (h - h \circ \varphi_i) \circ \varphi_x \longrightarrow 0$

$\left| \sum_{x \in \Lambda_n} (h - h \circ \varphi_i) \circ \varphi_x \right| \leq \|h\|_\infty |\partial \Lambda_n|$



$\checkmark \quad \forall g, h_1 \dots h_d \in L^1 \cap L^\infty$.

2)

A_n

3)

4)

$\frac{1}{|B_n(0)|}\sum_{x\in B_n(0)} |f|\circ\varphi_x$

d

$\|f\|_1$

...ed directly

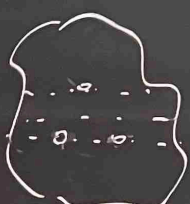
$\ldots_i - h_i\circ\varphi_i)$

φ_{e_i}

$\circ\varphi_i \quad i=1\ldots d$

0

$|\Lambda_n|$



2) $\{ f\in L^1 : \lim_{n\to\infty} \underbrace{\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n} f\circ\varphi_x}_{A_nf} \text{ exists a.e}\}$ is closed in L^1.

$$A_nf_k - (f-f_k)^* \le A_nf \le A_nf_k + (f-f_k)^*$$

$$\mu((f_n-f_k)^*>\varepsilon) \le \frac{C}{\varepsilon}\|f-f_k\|_1$$

3) $\mathcal{L} = \{ g + \sum_{i=1}^{d} (h_i - h_i\circ\varphi_i) : g, h_1 \ldots h_d \in L^1\cap L^\infty,\ g \text{ invariant} \}$ is dense in L^2. (Same as for Mean Ergodic Thm).

4) $f\in L^1\cap L^\infty$. Set $E_k := \{\limsup |A_nf| > 1/k\}$.

Then $\mu(E_k)<\infty$, E_k invariant under $\varphi_1\ldots\varphi_d$.

Then $f_k := f\mathbb{1}_{E_k}\in L^2$. Then $\exists f'\in\mathcal{L}$ s.t. $\|f_k - f'\|_2 < \varepsilon[1+\mu(E_k)]^{-1/2}$

Then $f_k' := f'\mathbb{1}_{E_k}$ obeys $\|f_k - f_k'\|_2 < \varepsilon[1+\mu(E_k)]^{-1/2}$.

CS: $\|f_k-f_k'\|_1 \le \mu(E_k)^{1/2}\|f_k-f_k'\|_2 < \varepsilon$.

Since $f_k'\in\mathcal{L}$. DCT $\Rightarrow$ $f\in$ set above. $\boxtimes$

Maximal function

Lemma Suppose $\forall f \in L^1$: $\mu(f^* > \lambda) \leq \frac{C}{\lambda} \|f\|_1$.
Let $p \in (1, \infty)$. Then $\forall f \in L^p$:

$$\|f^*\|_p \leq 2C^{1/p} \left(\frac{p}{p-1}\right)^{1/p} \|f\|_p$$

Pf: WLOG: $f \geq 0$. Set $g = f \mathbb{1}_{f \geq \alpha/2}$.
Then $f \leq g + \alpha/2$, so $f^* \leq g^* + \alpha/2$.

Hence $\mu(f^* > \alpha) \leq \mu(g^* > \alpha/2)$

$$\leq \frac{2C}{\alpha} \int g \, d\mu = \frac{2C}{\alpha} \int_{f > \alpha/2} f \, d\mu \quad (*)$$

$$\int f_*^p \, d\mu = \int_0^\infty p\, \alpha^{p-1} \mu(f_* > \alpha)\, d\alpha$$

$$\le 2Cp \int_0^\infty d\alpha \int d\mu \ \alpha^{p-2} f \, 1_{f \ge \alpha/2}$$

$$= 2C \frac{p}{p-1} \int d\mu \, (2f)^{p-1} f$$

$$= 2^p C \frac{p}{p-1} \int f^p \, d\mu \qquad \boxtimes$$

Thm (Wiener's Dominated Ergodic Thm) 1939

Suppose $\mu(X) < \infty$ and $f \in L^1 \log L = \{ f \in L^1 : \int |f| \log_+ |f| \, d\mu < \infty \}$.

Then $f^* \in L^1$

Pf $\int f^* d\mu \leq \underbrace{2\|f\|_1 \mu(\mathcal{X})}_{c} + \int_{2\|f\|_1}^{\infty} \mu(f_* > \alpha)\, d\alpha$

$\leq c + 2C \int_{2\|f\|_1}^{\infty} \int_{\mathcal{X}} \frac{1}{\alpha} |f| 1_{|f| > \alpha/2}\, d\mu\, d\alpha$

$= c + 2C \int |f| \log \frac{2|f|}{2\|f\|_1}\, d\mu < \infty$.

$\ldots/2$

$\ldots f|\log_+|f|\, d\mu < \infty \}$

<u>Note</u>. MusA fail when $\mu(\mathcal{X}) = \infty$.

(L^1-convergence fails in these cases).

- Converse holds in m.p. flows.

Thm (Banach '25)

Let $p \in [1, \infty)$, $\{T_n\}_{n \geq 1}$ bounded linear operators on L^p. Set

$$T^* f := \sup_{n \geq 1} |T_n f|$$

and assume $\forall f \in L^p$: $T^* f < \infty$ a.e.

Then $\{ f \in L^p : \lim_{n \to \infty} T_n f \text{ exists a.e.} \}$

is closed in L^p.

Pf Idea:

$$\sup_{f \in L^p} \mu(T^* f > \lambda \|f\|_p) \xrightarrow[\lambda \to \infty]{} 0$$