Harmonic Analysis of Markov chains

Thm (Pólya 1921) Simple symmetric random walk on $\mathbb{Z}^d$ is (is a Markov chain)

- recurrent in $d=1,2$
- transient in $d \geq 3$

Pf: Fourier analysis:

$$1_{\{S_n=0\}} = \int_{(-\pi,\pi)^d} \frac{dt}{(2\pi)^d} e^{it\cdot S_n}$$

Let $\{X_k\}_{k\geq 1}$ be iid, $\mathbb{Z}^d$-valued,

$$S_n := X_1 + \cdots + X_n$$

Then

$$P(S_n = 0) = E\int_{(-\pi,\pi)^d} \frac{dt}{(2\pi)^d}\, e^{i t\cdot \sum_{j=1}^{n} X_j}$$

$$= \int_{(-\pi,\pi)^d} \frac{dt}{(2\pi)^d}\, \varphi(t)^n \qquad (t\in\mathbb{R}^d)$$

where $\varphi(t) := E(e^{it\cdot X_1})$

$$\sum_{n=0}^{\infty} P(S_n=0) \overset{\text{MCT}}{=} \lim_{r\uparrow 1} \sum_{n=0}^{\infty} r^n P(S_n=0)$$

$$= \lim_{r\uparrow 1} \int_{(-\pi,\pi)^d} \frac{dt}{(2\pi)^d}\, \underbrace{\sum_{n=0}^{\infty} r^n \varphi(t)^n}_{\mathrm{Re}}$$

$$\sum_{n=0}^{\infty} P(S_n = 0) = \lim_{r \uparrow 1} \int_{(-\pi,\pi]^d} \frac{dt}{(2\pi)^d} \operatorname{Re} \frac{1}{1 - r\varphi(t)}$$

For SRW: $\varphi(t) = \frac{1}{2d} \sum_{j=1}^{d} \left(e^{it_j} + e^{-it_j}\right)$

$$= \frac{1}{d} \sum_{j=1}^{d} \cos(t_j)$$

$t \in (-\pi,\pi]^d$

$$1 - \varphi(t) = \frac{2}{d} \sum_{j=1}^{d} \sin(t_j/2)^2 \quad (= 0 \Leftrightarrow t = 0)$$

For $X_1 \overset{d}{=} -X_1$: $\varphi(t) \in \mathbb{R}$ and

$$\lim_{r \uparrow 1} \int_{(-\pi,\pi]^d} \frac{dt}{(2\pi)^d} \frac{1}{1 - r\varphi(t)} \overset{\substack{MCT \\ BCT}}{=} \int_{(-\pi,\pi]^d} \frac{dt}{(2\pi)^d} \frac{1}{1 - \varphi(t)}$$

$\in \mathbb{R}^d)$

For SRW

$$\left(\frac{2}{\pi}\right)^2 \frac{|t|^2}{2d} \le 1-\varphi(t) \le \frac{1}{2d}\sum_{j=1}^{d}|t_j|^2 = \frac{|t|^2}{2d}$$

$$\int_{|t|<\varepsilon} \frac{dt}{|t|^2} = \pi_{d-1}\int_0^{\varepsilon} r^{d-3}\,dr = \begin{cases} +\infty & d=1,2 \\ <\infty & d\ge 3. \end{cases}$$

Corollary Every symmetric RW with L^2-steps is recurrent in $d=1,2$.

Pf ① Under $X_1 \overset{d}{=} -X_1$:

$$1-\varphi(t) = E\left(1-\cos(tX)\right) \le \frac{1}{2}E\left((t\cdot X)^2\right) \le \frac{1}{2}|t|^2 E\left(|X|^2\right)$$

So $\displaystyle\int \frac{dt}{1-\varphi(t)} = +\infty$ in $d=1,2$.

Pf ②.

$$P(S_{2n}=0) = \sum_{x\in\mathbb{Z}^d} P(S_n=x)\,P(S_n=-x)$$

$$\overset{X_1 \overset{d}{=} -X_1}{=} \sum_{x\in\mathbb{Z}^d} P(S_n=x)^2$$

$$\geq \sum_{|x|_\infty\leq\sqrt{n}} P(S_n=x)^2$$

$$\geq \frac{\left[\sum_{|x|_\infty\leq\sqrt{n}} P(S_n=x)\right]^2}{\sum_{|x|_\infty\leq\sqrt{n}} 1}$$

$$\geq \frac{1}{n^{d/2}} \underbrace{P(|S_n|_\infty\leq\sqrt{n})^2}_{\geq c>0 \text{ by CLT}}$$

$\boxtimes$

$$\sum_{n\geq 1}^{\infty} \frac{1}{n^{d/2}} = +\infty \quad d=1,2$$

<u>Thm</u> (Chung–Fuchs) Let $X_1, X_2 \ldots$ be iid, $\mathbb{Z}^d$-valued and $S_n = X_1 + \cdots + X_n$. Then

$$\left.\begin{array}{ll} \boxed{d=1} & \dfrac{S_n}{n} \xrightarrow{P} 0 \\ & \quad\text{(WLLN)} \\ \boxed{d=2} & \dfrac{S_n}{n} \xrightarrow{w} N(0,\sigma^2) \end{array}\right\} \Rightarrow S \text{ recurrent}$$

$\tilde{T}_x = \inf\{n \geq 0 : S_n = x\}.$

<u>Pf</u> Let $\Lambda \subseteq \mathbb{Z}^d$ finite.

$$\sum_{n=0}^{\infty} P(S_n \in \Lambda) = \sum_{x \in \Lambda} \sum_{n \geq 0} P(S_n = x)$$

$$= \sum_{x \in \Lambda} \sum_{n \geq 0} \sum_{\ell=0}^{n} P(\tilde{T}_x = \ell, \; S_n - S_\ell = 0)$$

indep.

$$= \sum_{x \in \Lambda} \sum_{n \geq 0} P(\tilde{T}_x < \infty) P(S_n = 0)$$

$$\leq |\Lambda| \sum_{n \geq 0} P(S_n = 0)$$

So

$$\sum_{n=0}^{\infty} P(S_n = 0) \geq \frac{1}{|\Lambda|} \sum_{n=0}^{\infty} P(S_n \in \Lambda)$$

$=x\}.$

$-S_\ell=0)$

$=0)$

$\boxed{d=1}$ $\Lambda_k := \{x : |x|_\infty \le k\}$

$$\frac{1}{|\Lambda_k|}\sum_{n=0}^{\infty} P(S_n\in\Lambda_k)$$

$$\ge \frac{1}{|\Lambda_k|}\sum_{n=k}^{\lfloor k/\varepsilon\rfloor} P\Big(\Big|\frac{S_n}{n}\Big| \le \underbrace{\frac{k}{n}}_{\ge\varepsilon}\Big)$$

$$\ge \frac{1}{2k+1}\Big(\Big\lfloor\frac{k}{\varepsilon}\Big\rfloor-k\Big)\underbrace{\inf_{n\ge k} P\Big(\frac{|S_n|}{n}\le\varepsilon\Big)}_{\xrightarrow[k\to\infty]{} 1}$$

$$\ge \frac{1}{4}\,\frac{1}{\varepsilon} \quad \text{once } k \text{ large.}$$

$$\underset{\varepsilon\downarrow 0}{\Rightarrow} \sum_{n=0}^{\infty} P(S_n=0) = +\infty.$$

d

$\frac{1}{|}$

S_n

$\boxed{d=2}$ $\Lambda_k := \{ x \in \mathbb{Z}^d : |x|_\infty \le k \}$.

$$\frac{1}{|\Lambda_k|} \sum_{n \ge 0} P(S_n \in \Lambda_k)$$

$$\ge \sum_{m=1}^{M} \frac{1}{(2k+1)^2} \sum_{n=(m-1)k^2}^{mk^2-1} P\Big(\Big|\frac{S_n}{\sqrt{n}}\Big|_\infty \le \underbrace{\frac{k}{\sqrt{n}}}_{\le \frac{1}{\sqrt{m}}} \Big)$$

$$\ge \sum_{m=1}^{M} \underbrace{\frac{k^2}{(2k+1)^2}}_{\ge \frac{1}{4}} \min_{(m-1)k^2 \le n \le m^2k^2} P\Big(\Big|\frac{S_n}{\sqrt{n}}\Big|_\infty \le \frac{1}{\sqrt{m}} \Big)$$

$$\downarrow$$

$$P\Big(|N(0,\sigma^2)| \le \frac{1}{\sqrt{m}} \Big) \ge \frac{c}{m}$$

$$\ge c \sum_{m=1}^{M} \frac{1}{m} \gtrsim c \log M \longrightarrow \infty$$

So RW is recurrent. $\boxtimes$

Thm Let $d \geq 3$. Then any $\mathbb{Z}^d$-valued "truly at least 3-dimensional" random walk is transient.

Pf (idea) Assuming symmetry & irreducibility

"truly d-dimensional" means

$$1 - \varphi(t) \geq c|t|^2 \quad t \in (-\pi,\pi)^d$$

for some $c > 0$.

$$\overset{\shortparallel}{E}(1-\cos(t\cdot X)) \geq E\left((1-\cos(t\cdot X))\,1_{|X| \leq 1/\delta}\right)$$

Connection to harmonic analysis

$$Ph(x) := \sum_{y \in S} P(x,y)\, h(y).$$

Def A function $h : S \to \mathbb{R}$ is P-harmonic if $\forall x \in S: \sum_{y \in S} P(x,y)|h(y)| < \infty$ and $Ph = h$ on S.

Lemma If h is P-harmonic, then $M_n := h(X_n)$ is a martingale (under P^x, for all $x \in S$).

Pf

$$E^x(M_{n+1} | \mathcal{F}_n) = E^x(h(X_{n+1}) | \mathcal{F}_n) \overset{MP}{=} \sum_{y \in S} h(y)\, P(X_n, y) \overset{Ph=h}{=} h(X_n) = M_n.$$

Thm If M.C. X is irreducible recurrent, then all bounded and all non-negative P-harmonic functions are constant

Pf $M_n = h(X_n)$ ... $<$ bounded / ≥ 0 martingale

Hence, $M_\infty := \lim_{n\to\infty} M_n$ exist P^x-a.s. $\forall x \in S$.

So $\lim_{n\to\infty} h(X_n) = M_\infty$ P^x-a.s.

Since y visited ∞-many times a.s.

we have $\forall y \in S$: $h(y) = M_\infty$. $\boxtimes$

Thm For SRW on $\mathbb{Z}^d$, $d \geq 1$,
bounded P-harmonic functions are constant.

Pf: $\exists$ coupling of two copies of SRW with finite coupling time.

$$|h(x)-h(y)| = |E^x(h(X_n)) - E^y(h(Y_n))|$$
$$= |E^{(x,y)}(h(X_n)-h(Y_n))|$$
$$\leq \|h\|_\infty P^{x,y}(T > n)$$

coupling time

$x \in S.$

$d=2$

$\frac{1}{|\Lambda_k}$

S_n