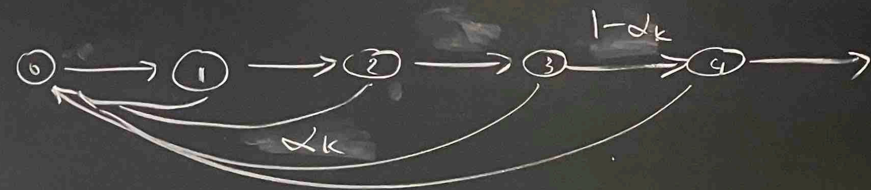
What goes wrong with transient chains

renewal chain



$(\alpha_k \in (0,1))$

claim: Transient $\iff \prod_{k\geq 0}(1-\alpha_k) > 0.$

Q: What does an invariant measure exist?

$$\nu(x) = \sum_{y\in S} \nu(y)\, P(y,x)$$

$$\left.\begin{aligned} \nu(k+1) &= \nu(k)(1-\alpha_k) \quad \forall k\geq 0 \\ \nu(0) &= \sum_{k\geq 1} \nu(k)\,\alpha_k \end{aligned}\right\} \quad \nu(k) = \nu(0)\prod_{j=0}^{k-1}(1-\alpha_j)$$

$$\nu(0) = \underbrace{\left[\sum_{k\geq 1} \alpha_k \prod_{j=0}^{k-1}(1-\alpha_j)\right]}_{\displaystyle \| \atop \displaystyle 1-\prod_{j\geq 0}(1-\alpha_j)} \nu(0) \quad \Rightarrow \quad \mathcal{I} \neq \emptyset$$

$\iff$ chain recurrent.

Convergence to equilibrium

Corollary (of Density of returns)

$\forall x,y \in S$:
$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P^k(x,y) = \frac{1}{E^y(T_y)}\, P^x(T_y<\infty)$$

invariant measure

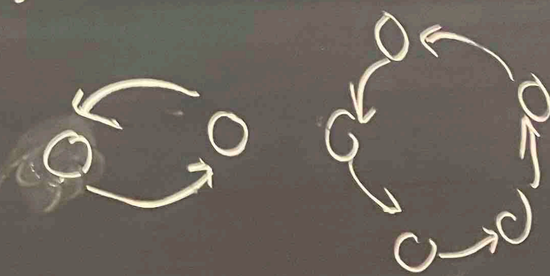
Pf We had $\displaystyle\lim_{n\to\infty} \frac{1}{n} N_n(y) = \frac{1}{E^y T_y} \mathbb{1}_{T_y<\infty}$, P^x-a.s.

$\frac{N_n(y)}{n} \le 1$ so just take expectations. $\boxtimes$

Note Only positively-recurrent chains have hope

Q: When does $P^n(x,y)$ itself converge?

Obstruction: Periodicity



Def $I_x := \{n \geq 1 : P^n(x,x) > 0\}$.

Note $m, n \in I_x \Rightarrow P^m(x,x) > 0 \wedge P^n(x,x) > 0$

$\Rightarrow P^{m+n}(x,x) > 0 \Rightarrow m+n \in I_x$

Def $d(x) := \gcd(I_x)$.

Lemma If M.C. irreducible then

$$\forall x, y \in S: \quad d(x) = d(y).$$

Pf: Let $x, y \in S$ $(x \neq y)$. Irreducibility $\Rightarrow \exists m, n \geq 1$

s.t. $P^m(y,x) > 0 \wedge P^n(x,y) > 0$.

$\Rightarrow m+n \in I_x \cap I_y$. So $d(x) \mid m+n$

Now take $r \in I_y$ and observe $P^{r+m+n}(x,x) \geq P^n(x,y) \underbrace{P^r(y,y)}_{>0} P^m(y,x)$

$\Rightarrow r+m+n \in I_x \Rightarrow d(x) \mid r \Rightarrow d(y) \geq d(x)$ $\boxtimes$

invariant measure

$P^x(T_y < \infty)$

(T_y)

$y < \infty$. P^x — a.s.

$\boxtimes$

we hope

?

Def

N.B.

Lemma

Pf (id

$\Rightarrow$

$\Rightarrow$

deta

Def. A state $x \in S$ is <u>aperiodic</u> if $d(x) = 1$.

N.B. Under irreducibility, whole M.C. aperiodic,
(if at least one state is)

Lemma Suppose $d(x) = 1$. Then

$$\exists n_0 \geq 0 \ \forall n \geq n_0(x) : n \in I_x .$$

Pf (idea) Show $\exists k \in I_x : (k+1) \in I_x$.

Then $\Rightarrow 2k, 2k+1, 2k+2 \in I_x$

$\Rightarrow 3k, 3k+1, 3k+2, 3k+3 \in I_x$.

nk, ..., nk+n = (n+1)k are in I_x
and so I_x contains all m > nk

details: textbook

Thm Consider an irreducible, positively recurrent, aperiodic MC. Let $\pi \in$ [illegible] unique inv. distribution.

Then $\forall_{x,y \in S}$: $P^n(x,y) \longrightarrow \pi(y)$.

(In fact: $\sum_{y \in S} \left| P^n(x,y) - \pi(y) \right| \longrightarrow 0$).

Proof by Coupling Define M.C. on $S \times S$ with tr. kernel:

$$\bar{P}((x_1, x_2), (y_1, y_2)) = \begin{cases} P(x_1, y_1) P(x_2, y_2) & x_1 \neq x_2 \\ P(x_1, y_1) \delta_{y_1, y_2} & x_1 = x_2 \end{cases}$$

Lemma $\bar{P}$ is transition kernel on $S \times S$. Moreover, $\forall \mu, \nu \in M_1(S)$, the M.C. initiated from $\mu \otimes \nu$ with tr. kernel $\bar{P}$ obeys:

$$\forall A \in \Sigma^{\otimes \mathbb{N}}: \quad \begin{aligned} \bar{P}^{\mu \otimes \nu}\left((X^1, X^2) \in A \times \Sigma^{\otimes \mathbb{N}}\right) &= P^{\mu}(X \in A) \\ \bar{P}^{\mu \otimes \nu}\left((X^1, X^2) \in \Sigma^{\otimes \mathbb{N}} \times A\right) &= P^{\nu}(X \in A) \end{aligned}$$

Pf $$\sum_{y_2} \bar{P}\big((x_1,x_2),(y_1,y_2)\big) = \begin{cases} P(x_1,y_1)\underbrace{\sum_{y_2} P(x_2,y_2)}_{=1} \\ P(x_1,y_1) \end{cases}$$

This + calculation proves the claim. $\boxtimes$

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coupling time

Lemma Let $T := \inf\{n \geq 0 : X^1_n = X^2_n\}$.

Then $\bar{P}^{\mu\otimes\nu}(T<\infty) = 1$ (under assumptions of Thm.)

Pf If $Q^{\mu\otimes\nu}$ = law of product chain,

then $Q^{\mu\otimes\nu}(T=n) = \bar{P}^{\mu\otimes\nu}(T=n)$, $n\in\mathbb{N}$.

ETS $Q^{\mu\otimes\nu}(T<\infty) = 1$

law of uncoupled chain

$y_2)$

$=1$

$\}$.

$n \in \mathbb{N}$,

<u>claim</u> Uncoupled chain is irreducible.

<u>Pf</u> Lemma + irreducibility

$\forall x_1, x_2, y_1, y_2 \ \exists\, n_0(x_1, y_1),\ n_0(x_2, y_2) \geq 1$

$: \forall n \geq n_0(x_1, y_1) : P^n(x_1, y_1) > 0$

$\forall n \geq n_0(x_2, y_2) : P^n(x_2, y_2) > 0$

$\Rightarrow \forall n \geq \max\{n_0(x_1, y_1), n_0(x_2, y_2)\}$

$P^n(x_1, y_1)\, P^n(x_2, y_2) > 0$

<u>claim</u> Uncoupled chain is (pos.) recurrent.

<u>Pf</u> $\pi \otimes \pi$ invariant distribution.

<u>claim</u> For any irreducible chain P

x recurrent $\Rightarrow \forall y \in S : P^x(T_y < \infty) = 1$

Let $x \in S$. Then $T \leq T_{(x,x)}$.

By claim: $Q^{\mu\otimes\nu}(T_{(x,x)} < \infty) = 1$. $\boxtimes$

Lemma (Coupling inequality). Let $\mu_n(\cdot) = P^{\mu}(X_n \in \cdot)$
$\nu_n(\cdot) = P^{\nu}(X_n \in \cdot)$.

Then $\|\mu_n - \nu_n\|_{TV} \leq \bar{P}^{\mu\otimes\nu}(T > n)$

Pf Pick $A \in S$. Then

$$\mu_n(A) - \nu_n(A) = E^{\mu\otimes\nu}\left(1_{X_n^1 \in A} - 1_{X_n^2 \in A}\right)$$

$$= E^{\mu\otimes\nu}\left(1_{T>n}\left(1_{X_n^1 \in A} - 1_{X_n^2 \in A}\right)\right)$$

$$\leq P^{\mu\otimes\nu}(T > n) \quad \boxtimes$$

$) = 1$

Pf of Thm. Let $\pi \in \mathcal{P}_1$. Note: $\pi_n = \pi$.

Let $x \in S$, $\mu = \delta_x$. Then $\mu_n(\cdot) = P^n(x, \cdot)$.

Note $\| P^n(x, \cdot) - \pi(\cdot) \|_{TV} = \frac{1}{2} \sum_{y \in S} | P^n(x,y) - \pi(y) |$

This is $\leq P^{\delta_x \otimes \pi}(T > n) \xrightarrow[n \to \infty]{} P^{\delta_x \otimes \pi}(T = \infty) = 0$.

So $\sum_{y \in S} | P^n(x,y) - \pi(y) | \xrightarrow[n \to \infty]{} 0$ $\boxtimes$