Recurrence, positive recurrence & inv. measures (S = countable)

Last time: $T_x := \inf\{n \geq 1 : X_n = x\}$. Say x is recurrent if $P^x(T_x < \infty) = 1$.

Lemma MC irreducible (i.e. $\forall x \neq y : P^x(T_y < \infty) > 0$) $\Rightarrow$

$$\exists x \in S : x\text{-recurrent} \Rightarrow \forall x \in S : x\text{-recurrent}.$$

Pf Suppose x recurrent. Irreducibility $\exists\, l, k \geq 0 : P^k(x,y) > 0 \wedge P^l(y,x) > 0$.

$$\underbrace{E^y(N(y))}_{=\sum_{k\geq 1} 1_{X_k = y}} = \sum_{n \geq 1} P^n(y,y) \geq \sum_{n\geq 1} P^{n+k+l}(y,y) \geq \sum_{n \geq 1} P^l(y,x) P^n(x,x) P^k(x,y)$$

$$= P^l(y,x) P^k(x,y) \underbrace{E^x(N(x))}_{= \infty \text{ if } x \text{ recurrent}} = +\infty.$$

Thm Let $x \in S$ be recurrent. Define $\nu_x(y) := E^x\left(\sum_{k=1}^{T_x} 1_{X_k = y}\right)$

Then $\nu_x(x) = 1 \;\wedge\; \forall y \in S : \nu_x(y) < \infty \;\wedge\; \nu_x \in \phi$.

If M.C. irreducible, then

$$\forall \mu \in \phi \;\; \forall y \in S : \mu(y) = \mu(x)\nu_x(y).$$

i.e., $\phi = \{\lambda \nu_x : \lambda > 0\}$.

Pf $\boxed{\nu_x \in \mathcal{I}}$ We have $\nu_x(y) = \sum_{n\geq 1} P^x(X_n = y, T_x \geq n)$ $\quad \mathcal{F}_n := \sigma(X_0, \dots, X_n)$

$$\overset{\text{Tonelli}}{=} \sum_{z\in S}\sum_{n\geq 1} P^x(X_n = y, X_{n-1} = z, \underbrace{T_x \geq n}_{\in \mathcal{F}_{n-1}})$$

$$= \sum_{z\in S}\sum_{n\geq 1} E^x\left(1_{X_{n-1}=z} 1_{T_x\geq n} E^x(1_{X_n=y} \mid \mathcal{F}_{n-1})\right)$$

$$\overset{MP}{=} \sum_{z\in S}\sum_{n\geq 1} E^x\left(1_{X_{n-1}=z} 1_{T_x\geq n} P(X_{n-1}, y)\right)$$

$$= \sum_{z\in S} P(z,y) \sum_{k\geq 0} P^x(X_k = z, T_x > k) = \sum_{z\in S} P(z,y) \underbrace{E^x\left(\sum_{k=0}^{T_x-1} 1_{X_k=z}\right)}_{=\nu_x(z)}$$

$\forall n \geq 1$

So $\nu_x(y) = \sum_{z\in S} \nu_x(z) P^n(z,y)$

Now $\forall z\in S$: either $P^z(T_x<\infty)=0$ or $P^z(T_x<\infty)>0$

if $P^z(T_x<\infty)=0 \Rightarrow P^x(T_z<\infty)=0$ for otherwise $P^x(N(x)<\infty)>0$.

$\Rightarrow \nu_x(z)=0$.

if $P^z(T_x<\infty)>0 \Rightarrow \exists n\geq 1 : P^n(z,x)>0$.

$\Rightarrow 1 = \nu_x(x) = \sum_{y\in S}\nu_x(y)P^n(y,x) \geq \nu_x(z)P^n(z,x) \Rightarrow \nu_x(z)<\infty$.

Pf of $\boxed{\mu(\cdot) = \mu(x)\,\nu_x(\cdot)}$ Assume irreducibility, $\mu \in \phi$.

Then
$$\mu(y) = \sum_{z\in S} \mu(z)\,P(z,y) = \mu(x)\,P(x,y) + \sum_{z\neq x} \mu(z)\,P(z,y)$$
$$= \mu(x)\,P(x,y) + \sum_{z_1\neq x} \mu(x)\,P(x,z_1)\,P(z_1,y) + \sum_{z_0,z_1\neq x} \mu(z_0)\,P(z_0,z_1)\,P(z_1,y)$$
$$\vdots$$
$$= \mu(x)\Big[P(x,y) + \underbrace{\sum_{k=1}^{n}\ \sum_{\substack{z_1\dots z_k\neq x\\ z_0=x,\ z_{k+1}=y}}\ \prod_{i=1}^{k+1} P(z_{i-1},z_i)}_{\sum_{k=1}^{n} P^x(X_k=y,\ T_x\geq k)}\Big] + \underbrace{\sum_{\substack{z_0\dots z_n\neq x\\ z_{n+1}=y}} \mu(z_0)\prod_{i=1}^{n+1} P(z_{i-1},z_i)}_{\geq 0}$$

So taking $\geq$ & $n\to\infty$ we get
$$\mu(y) \geq \mu(x)\,\nu_x(y)$$
Then $\forall n\geq 1$:
$$\mu(x) = \sum_{y\in S}\mu(y)\,P^n(y,x) \geq \mu(x)\sum_{y\in S}\nu_x(y)\,P^n(y,x) \overset{\nu_x\in\phi}{=} \mu(x)\,\nu_x(x) = \mu(x).$$
So equality holds $\forall y\in S$ s.t. $P^n(y,x)>0$. By irreducibility $\forall y\in S$: $\mu(y) = \mu(x)\,\nu_x(y)$.

$\boxtimes$

Note $\nu_x(S) = \sum_{y\in S} \nu_x(y) = \sum_{y\in S} E^x\left(\sum_{k=1}^{T_x} 1_{X_k=y}\right) = E^x T_x$.

Def: x is positive recurrent if $E^x T_x < \infty$

x is null recurrent if $P^x(T_x<\infty)=1 \wedge E^x T_x = \infty$.

x is transient if $P^x(T_x<\infty)<1$

Corollary Suppose $\mu \in \mathscr{I}$ is s.t. $\mu(S)<\infty$. Then

$\forall x \in S$: $\mu(x)>0 \Rightarrow x$ positive recurrent.

If M.C. irreducible, then $\mu(x) = \dfrac{\mu(S)}{E^x T_x}$.

Pf: $\mu(x)>0 \Rightarrow$ $\infty = \sum_{n\ge 1}\mu(x) = \sum_{n\ge1}\sum_{y\in S}\mu(y)P^n(y,x) = \sum_{y\in S}\mu(y)E^y(N(x)) \le \mu(S)\dfrac{1}{1-P^x(T_x<\infty)}$

$E^y(N(x)) = \dfrac{P^y(T_x<\infty)}{1-P^x(T_x<\infty)} \le \dfrac{1}{1-P^x(T_x<\infty)}$

$\Rightarrow x$ recurrent.

Previous proof: $\mu(y) \ge \mu(x)\nu_x(y) \Rightarrow \mu(S) \ge \mu(x)\nu_x(S) = \mu(x)E^x T_x$

$\Rightarrow x$ is positive recurrent

When M.C. irreducible, = holds. $\boxtimes$

Corollary: For irreducible M.C., TFAE:

(1) $\exists x \in S$: positive recurrent

(2) $\exists \mu \in \phi$: $\mu(S) < \infty$

(3) $\forall x \in S$: positive recurrent.

Pf: (1) $\Rightarrow$ (2) $\nu_x(S) = E^x T_x < \infty$.

Rest by Corollary.

Thm (Density of returns) Let $N_n(y) = \sum_{k=1}^{n} 1_{X_k = y}$. Then

$$\forall x, y \in S: \; (y \text{ recurrent } \Rightarrow) \; \lim_{n \to \infty} \frac{N_n(y)}{n} = \frac{1}{E^y T_y} 1_{T_y < \infty} \quad P^x\text{-a.s.}$$

(Kac recurrence)

Pf by Renewal thm applied to $\{T_y^{j+1} - T_y^{j}\}_{j \geq 0}$.

Thm (Ratio limit thm) Let M.C. be irreducible. Let x = recurrent.

Then $\forall y, z \in S$: $$\lim_{n \to \infty} \frac{N_n(y)}{N_n(z)} = \frac{\nu_x(y)}{\nu_x(z)} \quad P^x\text{-a.s.}$$

(Hopf ratio ergodic thm).

When M.C. irreducible, $=$ holds. $\triangle$