Invariant measures

$\mathcal{I} = \{\mu \in \mathcal{M}(S) : \text{invariant}, \neq 0\}$.

Remark: M.C. started from μ defined m.p.s. called "Markov shift".

As a consequence: $\forall f \in L^1(\mu)$:

$$\frac{1}{n}\sum_{k=0}^{n-1} f(X_k) \xrightarrow[n\to\infty]{} \bar{f} \qquad \mu\text{-a.e.}$$

+ ergodicity (under conditions).

Also: reversible measure $\longleftrightarrow$ chain run backwards has same law.

$q(x, A)$ = reverse transition probability

$$\int_A \mu(dx)\, p(x, B) = \int_B \mu(dy)\, q(y, A). \quad *$$

Functional-analytic connection:

Thm Let (S,Σ) = Standard Borel, and $\mu \in \mathcal{P}$ (σ-finite).
Then $Pf(x) := \int p(x,dy) f(y) = E^x(f(X_1))$
defines a bounded linear operator $L^2(\mu) \to L^2(\mu)$.
Moreover, P^+ takes the form

$$P^+ f(x) = \int q(x,dy) f(y).$$

and so

$$\mu\text{-reversable} \Leftrightarrow P^+ = P$$

$$(P^+ g, f) = (g, Pf)$$

Pf: Jensen: $\forall p \geq 1$

$$|Pf(x)|^p = \left| \int p(x,dy) f(y) \right|^p \leq \int p(x,dy) |f(y)|^p$$

So $\forall p \geq 1$:

$$\int d\mu \, |Pf|^p \leq \int d\mu \, P|f|^p \underset{\mu \in \Phi}{=} \int d\mu \, |f|^p$$

So $\|Pf\|_p \leq \|f\|_p \quad p \in [1, \infty]$.

For adjoint

$$(g, Pf) = \int \mu(dx)\, g(x)\, p(x,dy)\, f(y)$$

$$\overset{(*)}{=} \int \mu(dy)\, q(y,dx)\, g(x)\, f(y)$$

$$= (Qg, f) \quad \ldots \quad P^\dagger = Q.$$

<u>Note</u> Functional analytic picture exists regardless of regularity of $\mathbb{S}$.

<u>Caveat</u> Need $\mu \in \mathcal{P}$!?

<u>Resolved</u> by working on $C_c(\mathbb{S})$.

↳ key for theory of general Markov processes

Moving back to existence of invariant measures:

<u>Thm</u> (Markov/strong Markov property). Let $\{X_n\}_{n\ge 0} = MC$ on $\mathbb{S}$.

<u>Denote</u> $\mathcal{F}_n := \sigma(X_0, \dots, X_n)$. Then

$\forall n \ge 0 \; \forall A \in \Sigma^{\otimes \mathbb{N}}$:

$$P((X_n, X_{n+1}, \dots) \in A \mid \mathcal{F}_n) = P^{X_n}(X \in A) \quad \text{a.s.}$$

(Markov property)

- For any stopping time T for $\{\mathcal{F}_n\}_{n\geq 0}$

$$\forall A \in \mathcal{F}_T: \quad P\left(T<\infty \wedge (X_T, X_{T+1}, \dots) \in A \,\middle|\, \mathcal{F}_T\right)$$

$$= P^{X_T}(X \in A) \quad \text{a.s. on } \{T<\infty\}.$$

(strong Markov property)

<u>Recall</u> T stopping time: $\forall n \geq 0: \{T \leq n\} \in \mathcal{F}_n$

$$\mathcal{F}_T := \{A \in \Sigma^{\otimes \mathbb{N}} : (\forall n \geq 0 : A \cap \{T \leq n\} \in \mathcal{F}_n)\},$$

<u>Pf of MP</u>: Realize the chain on $(S^{\mathbb{N}}, \Sigma^{\otimes \mathbb{N}})$. θ = left shift.

Pick $A = A_0 \times \cdots \times A_m \times S \times S \times \cdots$, $B = B_0 \times \cdots \times B_n \times S \times \cdots \in \mathcal{F}_n$

$$E^{\mu}\left(1_B 1_{(X_n, X_{n+1}, \ldots) \in A}\right) = E\left(1_B 1_A \circ \theta^n\right)$$

$\mu(\cdot) = P(X_0 \in \cdot)$

$$= \int_{B_0 \times \cdots \times B_{n-1} \times (B_n \cap A_0) \times A_1 \cdots \times A_m} \mu(dx_0) \prod_{i=1}^{n} p(x_{i-1}, dx_i) \underbrace{\prod_{j=n+1}^{n+m} p(x_{j-1}, dx_j)}$$

$$= \int_{B_0 \times \cdots \times B_n} \mu(dx_0) \prod_{i=1}^{n} p(x_{i-1}, dx_i)\, P^{x_n}(X \in A)$$

$$= E\left(1_B P^{X_n}(X \in A)\right).$$

<u>claim</u> $x \longmapsto P^x(X \in A)$ Σ-measurable $\forall A \in \Sigma^{\otimes \mathbb{N}}$. ✓ TRUE for semialgebra of product events

π-$\lambda \Rightarrow$ TRUE $\forall A \in \Sigma^{\otimes \mathbb{N}}$.

<u>claim</u> $E^{\mu}(1_B 1_A \circ \theta^n) = E^{\mu}(1_B P^{X_n}(X \in A))$

holds $\forall A \in \Sigma^{\otimes \mathbb{N}}$ $\forall B \in \mathcal{F}_n$. by —— || ——

So by def of cond. expectation:

$$E\left(\mathbb{1}_A \circ \theta^n \mid \mathcal{F}_n\right) = P^{X_n}(X \in A) \quad \text{a.s.}$$

$\nwarrow$ $\mathcal{F}_n$-meas.

Proof of SMP Let $B \in \mathcal{F}_T$. Then

$$E^\mu\left(\mathbb{1}_{B \cap \{T<\infty\}} \mathbb{1}_A \circ \theta^T\right)$$

$$= \sum_{n=0}^{\infty} E^\mu\left(\mathbb{1}_{\underbrace{B \cap \{T=n\}}_{\in \mathcal{F}_n}} \mathbb{1}_A \circ \theta^n\right)$$

$$\overset{\text{MP}}{=} \sum_{n=0}^{\infty} E^\mu\left(\mathbb{1}_{B \cap \{T=n\}} P^{X_n}(X \in A)\right)$$

$$= E^\mu\left(\mathbb{1}_{B \cap \{T<\infty\}} P^{X_T}(X \in A)\right)$$

Now uniqueness argument from def. of cond. exp to conclude. $\boxtimes$

a.s.,

as.

MP says: MC after X_n revealed
$\approx$ MC started for X_n.

SMP says: Also TRUE for stopping times.

Idea: use a recurrent state to
construct invariant measure

Def A state $x \in S$ is said
to be recurrent if
$$P^x(T_x < \infty) = 1$$

for $T_x := \inf\{n \geq 1 : X_n = x\}$.

Lemma Let $N(x) = \sum_{k=1}^{\infty} 1_{X_k = x}$. Then TFAE:

1) $P^x(T_x < \infty) = 1$

2) $P^x(N(x) = +\infty) = 1$

3) $E^x(N(x)) = \infty$

Pf $\boxed{(1) \Rightarrow (2)}$ Define iterates of T_x by

$$T_x^1 := T_x, \quad T_x^{j+1} := \inf\{n > T_x^j : X_n = x\}.$$

(stopping times)

SMP $P(T_x^{j+1} - T_x^j = n \mid \mathcal{F}_{T_x^j})$

$\overset{\text{SMP}}{=} P^x(T_x = n)$ on $\{T_x^j < \infty\}$.

So under x-recurrent $\Rightarrow$ $\{T_x^{j+1} - T_x^j\}_{j \geq 0} \overset{\text{iid}}{\sim} T_x$.

So $P^x(T_x<\infty)=1 \Rightarrow P^x(\forall j\ge 0: T_x^{j+1}-T_x^j<\infty)=1$

and $N(x)=\sum_{j\ge 1} 1_{\{T_x^j<\infty\}} = +\infty \quad P^x\text{-a.s.}$

Pf $(3)\Rightarrow(2)$:

$$E^y(N(x)) = P^y(T_x<\infty)\sum_{k\ge 0} P^x(T_x<\infty)^k$$

$$= \begin{cases} \dfrac{P^y(T_x<\infty)}{1-P^x(T_x<\infty)} \\ =\infty \text{ if } y=x,\ P^x(T_x<\infty)=1. \end{cases}$$