Markov chains — reversible measures

Last time M.C. with tr. prob. p: μ = <u>invariant</u> if $\mu P = \mu$

$$\mu(A) = \int \mu(dx)\, p(x, A)$$

<u>Note</u> $\Rightarrow \forall n \geq 1: \mu P^n = \mu$.

Non-uniqueness can be caused by states "not communicating".

Def Let S = countable. We say transition kernel $P: S \times S \to [0,1]$ is <u>irreducible</u> if $\forall x, y \in S \; \exists n \geq 0: P^n(x,y) > 0$.

Lemma If P is irreducible and $\nu \in \mathcal{I} \Rightarrow \forall x \in S: \nu(x) > 0$.

Pf Let $x, y \in S$, $n \geq 0$ s.t. $P^n(x,y) > 0$. Then

$$\nu(y) = \sum_{z \in S} \nu(z) P^n(z,y) \geq \nu(x) P^n(x,y) \quad \text{so } \nu(x) > 0 \Rightarrow \nu(y) > 0;$$

Lemma Let (S, Σ) = standard Borel space, p = tr. prob.
Let $\mu \in \mathcal{P}_1$. Then $\exists \{X_n\}_{n\in\mathbb{Z}}$ s.t.

$\forall n \in \mathbb{Z}$: $P(X_n \in \cdot) = \mu(\cdot)$

$\wedge$ $P(X_{n+1} \in \cdot \mid \sigma(X_k : k \le n)) = p(X_n, \cdot)$ a.s.

Pf. Realize X_n as coordinate projection on $(S^{\mathbb{Z}}, \Sigma^{\otimes \mathbb{Z}})$

Define $\mathcal{F}_n := \sigma(X_k : -n \le k \le n)$.

Let $P_n(A_{-n} \times \cdots \times A_n) = \int_{A_{-n}\times\cdots\times A_n} \mu(dx_{-n})\, p(x_{-n}, dx_{-n+1}) \cdots p(x_{n-1}, dx_n)$

By μ stationary $\{P_n\}$ consistent family. Kolmogorov Extension Thm $\Rightarrow$ $\{P_n\}$ are restrictions of unique prob. measure on $(S^{\mathbb{Z}}, \Sigma^{\otimes\mathbb{Z}})$.

tr. prob:

$= p(X_n, \cdot)$ a.s.

$(S^{\mathbb{Z}}, \Sigma^{\otimes \mathbb{Z}})$

$\ldots_{-n+1})$

$\ldots dx_n)$

$\ldots_n\}$

$\ldots (S^{\mathbb{Z}}, \Sigma^{\otimes \mathbb{Z}})$;

Q: What's the laws of $\{X_{-n}\}_{n \in \mathbb{Z}}$?

Lemma Let S = contable, $\mu \in \mathcal{P}_1$, $\{X_n\}$ as above.

The $\{X_{-n}\}_{n \in \mathbb{Z}}$ is also a MC with initial distribution μ and transition kenel

$$Q(x,y) = \begin{cases} \dfrac{\mu(y) P(y,x)}{\mu(x)} & (\mu(x) > 0) \\ \delta_{xy} & \mu(x) = 0. \end{cases}$$

Pf: $P(X_0 = y_n, \ldots, X_n = y_0) = \mu(y_n) \prod_{i=1}^{n} P(y_{n-i+1}, y_{n-i})$

$= \frac{\mu(y_n)}{\mu(y_{n-1})} P(y_n, y_{n-1}) \frac{\mu(y_{n-1})}{\mu(y_{n-2})} P(y_{n-1}, y_{n-2}) \ldots \frac{\mu(y_1)}{\mu(y_0)} P(y_1, y_0) \mu(y_0)$

$= Q(y_{n-1}, y_n) Q(y_{n-2}, y_{n-1}) \cdots Q(y_0, y_1) \mu(y_0)$

W

Q

Le

Fo

S

M

M

t

We call the reversal the "reversed chain".

$\underline{Q}$: What if S not countable.

$\underline{\text{Lemma}}$ Let (S, Σ) = standard Borel, p = trans. prob.

For each $\mu \in \mathcal{P}$, $\exists\, q$ = transition probability

s.t. $\forall A, B \in \Sigma$, $\int_A \mu(dx)\, p(x, B) = \int_B \mu(dy)\, q(y, A)$.

Moreover, if $\{X_n\}_{n \in \mathbb{Z}}$ is MC. with initial distribution μ and tr. prob. p, then $\{X_{-n}\}_{n \in \mathbb{Z}}$ is M.C. with transition prob. q.

$\square$

(y_0)

$) \mu(y_0)$

Pf: Define prob. measure ν by

$$\nu(A \times B) = \int_A \mu(dx)\, p(x,B)$$

Note $\nu(A \times S) = \int_A \mu(dx) = \mu(A)$

$$\nu(S \times B) = \int \mu(dx)\, p(x,B) = \mu(B)$$

(S,Σ) = standard Borel $\Rightarrow \exists$ reg. conditional prob. given $\mathcal{G} = \{ S \times B : B \in \Sigma \}$.

Denoting this RCP by q we have

$$\int_B q(y,A)\, \mu(dy) = \nu(A \times B) = \text{above}$$

Now check that this gives reversed MC.

Def We say that (σ-finite) μ is reversible for transition probability p if

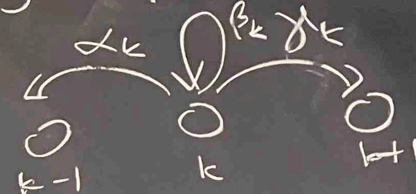
S = countable $\Rightarrow$ $\forall x, y \in S$: $\mu(x) P(x,y) = \mu(y) P(y,x)$

S = standard Borel $\Rightarrow$ $\forall A, B \in \Sigma$: $\int_A \mu(dx) p(x,B) = \int_B \mu(dy) p(x,A)$

Lemma Every reversible measure is invariant.

Pf $\int \mu(x) p(x,B) = \int_B \mu(dx) p(x,S) = \int_B \mu(dx) = \mu(B)$.

Ex <u>Birth-death chain</u> $S = \mathbb{N}$



<u>Need</u> $\nu(k)\, P(k,k+1)$
$= \nu(k+1)\, P(k+1,k)$

$$\nu(k)\,\gamma_k = \nu(k+1)\,\alpha_{k+1}$$

$$\nu(n) := \prod_{k=1}^{n} \frac{\gamma_{k-1}}{\alpha_k}$$ defines reversible measure (assuming $\alpha_k > 0$)

Ex <u>Random walk on weighted graph.</u>

$$P(x,y) := \frac{w(x,y)}{\pi(x)} \qquad \pi(x) = \sum_{z:\, z\sim x} w(x,z) > 0$$

Then $\pi(x)\,P(x,y) = w(x,y)$

so if $w(x,y) = w(y,x) \Rightarrow \pi$ is reversible measure

Ex: Biased random walk on $\mathbb{Z}$

$$p(x, x+1) = p, \quad p(x, x-1) = 1-p.$$

$$p \in (0,1),$$

$\nu(k) := 1$ is invariant

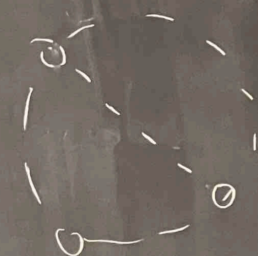
is NOT reversible unless $p = 1-p = 1/2$.

$$\nu(k) := \left(\frac{p}{1-p}\right)^k \text{ is reversible.}$$

Q: What conditions ensure existence of reversible measures?

Thm (Kolmogorov cycle conditions). Let S = countable and P = transition kernel on S. Assume P irreducible.

Then there exists a reversible measure if and only if



$$\forall n \geq 1 \ \forall x_0, x_1, \ldots, x_n = x_0 \in S:$$

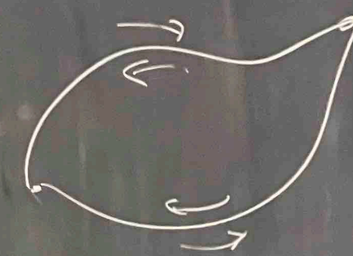
$$\prod_{i=1}^{n} P(x_{i-1}, x_i) = \prod_{i=1}^{n} P(x_{n-i+1}, x_{n-i})$$

Pf ν reversible $\Rightarrow$ $\nu(x_0)\,\text{LHS} = \nu(x_n)\,\text{RHS}$

$\boxed{\Rightarrow}$ Since $\nu(x_0) = \nu(x_n) > 0 \Rightarrow \text{LHS} = \text{RHS}$.

$\boxed{\text{Pf of } \Leftarrow}$ Pick $x_0 \in S$, set $\nu(x_0) = 1$

and, given any path $x_0, \dots, x_n$ with $x_n = x$



set:

$$\nu(x) := \prod_{i=1}^{n} \frac{P(x_{i-1}, x_i)}{P(x_i, x_{i-1})}$$

Need to check: (1) $\exists$ path where RHS meaningful
(implied by irreducibility)

(2) definition does NOT depend on path
(implied by cycle conditions) $\boxtimes$