Markov chains $\quad (S, \Sigma) = $ state space

- $\{X_n\}_{n\geq 0}$: $\quad P(X_{n+1} \in B \mid \sigma(X_k : k \leq n)) = P(X_{n+1} \in B \mid \sigma(X_n))$
- transition probability $\quad x, B \longmapsto p(x, B)$ use for this.
- For $\mu \in \mathcal{M}_1(S)$ $\exists \{X_n\}_{n\geq 0}$ s.t.

  1) $P^\mu(X_0 \in \cdot) = \mu(\cdot)$

  2) $\forall n \in \mathbb{Z}$: $P^\mu(X_{n+1} \in \cdot \mid \underbrace{\sigma(X_k : k \leq n)}_{=: \mathcal{F}_n}) = p(X_n, \cdot)$ a.s.

  no dep. on n "time homogeneous" case.

  P^μ law of MC with initial distribution μ

  if $\mu = \delta_x$ ... $P^x := P^{\delta_x}$

- transition kernel $\quad P(x,y) = \dfrac{dp(x, \cdot)}{dm(\cdot)}$ where m is "suitable" measure on S.

S = countable $\Rightarrow$ m = counting measure.

Q: What's the law of X_n? Convergence? Characterization of the limit?

Notation $p^{n+1}(x,B) := \int p^n(x,dz)\,p(z,B)$ defined recursively

$p^1(x,\cdot) = p(x,\cdot)$

$$\mu P(B) := \int \mu(dx)\,p(x,B).$$

Lemma $\forall n \geq 0$: $P^\mu(X_n \in \cdot) = \mu P^n(\cdot)$

Pf $P^\mu(X_{n+1} \in B) = E^\mu\left(E\left(\mathbb{1}_{X_{n+1}\in B} \mid \mathcal{F}_n\right)\right)$

$= E^\mu\left(p(X_n, B)\right)$

$= \int \mu P^n(dx)\,p(x,B) = \mu P^{n+1}(B)$

$\uparrow$ induction

Suppose μP^n converges. Then also μP^{n+1} converges. So limit has to obey $\nu P = \nu$.

<u>Def</u> We say a <u>measure</u> ν on (S, Σ) is <u>invariant</u>/<u>stationary</u> if $\nu P = \nu$

i.e. $\forall B \in \Sigma: \ \nu(B) = \int \nu(dx)\, p(x, B)$

$\mathcal{I} := \{ \nu : \text{invariant} \wedge \nu \neq 0 \}$.

$\mathcal{I}_1 := \{ \nu \in \mathcal{I} : \nu(S) = 1 \}$. <u>invariant distributions</u>

Q: Existence & uniqueness.

$\underline{\text{Lemma}}$ S finite $\Rightarrow \mathcal{I}_1 \neq \emptyset$ (Perron-Frobenius)

Pf Want ν: $\nu(x) = \sum_{y \in S} \nu(y) P(y,x) \quad \forall x \in S$.

So ν left eigenvector with eigenvalue 1 of P

P stochastic, $\mathbb{1}$ is right eigenvector w/ eigenvalue 1.

So $\exists f \in \mathbb{R}^S$ s.t. $fP = f$ ... $f(x) = \sum_y f(y) P(y,x)$

$\underline{\text{claim}}$ $|f| P = |f|$ (because $P(\cdot,\cdot) \geq 0$)

Pf $\forall x \in S$: $|f(x)| \leq \sum_{y \in S} |f(y)| P(y,x)$ $(*)$.

$$(|f|, 1) \leq \sum_{x,y \in S} |f(y)| P(y,x) \mathbb{1}(x) = (|f|, P1)$$
$$= (|f|, 1)$$

$\Rightarrow$ equality holds in $(*)$

$\Rightarrow |f|$ is an invariant measure, normalize to get $\nu \in \mathcal{I}_1$ $\boxtimes$

Thm Suppose S = compact metric space. Assume also that $f \longrightarrow Pf$ maps $C(S)$ to $C(S)$. (Krylov-Bogolyubov '37)

Then $\phi_1 \neq \emptyset$

$Pf(x) := \int p(x,dz) f(z)$

Pf: Let $f: S \to \mathbb{R}$ be continuous. Define

$$\phi_n(f) := E^{\mu}\left(\frac{1}{n}\sum_{k=0}^{n-1} f(X_k)\right)$$

Note $|\phi_n(f)| \leq \|f\| := \sup_{x \in S} |f(x)|$ ... $\phi_n \in C(S)^*$.

$C(S)$ separable + ϕ_n equicontinuous $\Longrightarrow$

$\exists\, n_k \to \infty$ $\forall f \in C(S)$: $\phi(f) := \lim_{k\to\infty} \phi_{n_k}(f)$

Note: $|\phi(f)| \leq \|f\|$, $f \geq 0 \Rightarrow \phi(f) \geq 0$, $\phi(1) = 1$.

Riesz Rep theorem: $\exists\, \nu \in \mathcal{M}_1$ $\forall f \in C(S)$: $\phi(f) = \int \nu(dx) f(x)$

Note: $\phi_n(Pf) - \phi_n(f) = \frac{1}{n}\sum_{k=0}^{n-1}\left[\mu P^k(Pf) - \mu P^k f\right]$

$= \frac{1}{n}\left(\mu P^n f - \mu f\right) \longrightarrow 0$.

So $\int \nu Pf = \int \nu f \quad f \in C(S)$.

Take $f \uparrow 1_O \quad O$ - open

π/λ-Thm $\Rightarrow \nu P = \nu$. $\boxtimes$

Ex • Ehrenfest chain $S = \{0, \dots, n\}$, $p(k,k+1) = \frac{n-k}{n}$, $p(k,k-1) = \frac{k}{n}$.

claim $\nu(k) = \binom{n}{k} 2^{-k}$ is invariant.

Pf $\nu(k-1)\, p(k-1,k) + \nu(k+1)\, p(k+1,k)$

$$= 2^{-n}\left[\binom{n}{k-1}\frac{n-k+1}{n} + \binom{n}{k+1}\frac{k+1}{n}\right]$$

$$= 2^{n}\binom{n}{k}\left[\frac{k}{n-k+1}\,\frac{n-k+1}{n} + \frac{n-k}{k+1}\,\frac{k+1}{n}\right] = 2^{-n}\binom{n}{k}.$$

• card shuffle $g \in \mathcal{M}_1(\mathcal{S}_n)$, $p(\sigma,\tilde\sigma) = g(\sigma^{-1}\tilde\sigma)$

claim $\nu(\sigma) := \frac{1}{n!}$ invariant

Pf $\sum_{\tilde\sigma} \nu(\tilde\sigma)\, p(\tilde\sigma,\sigma) = \frac{1}{n!}\sum_{\tilde\sigma} g(\tilde\sigma^{-1}\sigma) = \frac{1}{n!}\sum_{\tilde\sigma} g(\tilde\sigma) = \frac{1}{n!}$

Biased SRW

$$\overset{1-p}{\curvearrowleft}\ 0\ \overset{p}{\curvearrowright}$$

$$S = \mathbb{Z}, \quad p(x, x+1) = p, \quad p(x, x-1) = 1-p$$

$$\nu(k-1)\,p(k-1,k) + \nu(k+1)\,p(k+1,k) = \nu(k)$$

$$\Leftrightarrow \quad \nu(k-1)\,p + \nu(k+1)(1-p) = \nu(k)$$

$$= \nu(k)p + \nu(k)(1-p)$$

$$\Leftrightarrow \quad \nu(k+1) - \nu(k) = \frac{p}{1-p}\left[\nu(k) - \nu(k-1)\right]$$

$$\nu(k) = \alpha + \beta\left(\frac{p}{1-p}\right)^k$$

$p \neq \frac{1}{2}$: Two linearly indep. solutions! $(\mathcal{I}_1 \neq \emptyset)$

$(\dim \mathcal{I} \geq 2)$

- Dull chain $S = \mathbb{Z}$

$$p(k, k+1) = 1, \quad \forall k \geq 0.$$

$$\nu(k) = \nu(k-1)\, p(k-1, k) \quad k \geq 1$$

$$\nu(0) = 0$$

$$\Rightarrow \nu(k) = 0 \quad \forall k \geq 0$$

$$\Rightarrow \mathcal{I} = \phi.$$