Markov chains

Def A sequence of (S, Σ)-valued RV's is Markov chain for filtration $\{\mathcal{F}_n\}_{n \geq 0}$ if

1) $\forall n \geq 0$: X_n is $\mathcal{F}_n$-measurable
2) $\forall n \geq 0$ $\forall B \in \Sigma$:

$$P(X_{n+1} \in B \mid \mathcal{F}_n) = P(X_{n+1} \in B \mid \sigma(X_n))$$

Doob-Dynkin: $P(X_{n+1} \in B \mid \sigma(X_n)) = h(X_n)$ for measurable $h: S \to [0,1]$

write this as $p_n(X_n, B)$

Def A map $p: S \times \Sigma \to [0,1]$ is transition probability (a.k.a.) Markov kernel if

(1) $\forall B \in \Sigma$: $x \longmapsto p(x, B)$ is Σ-measurable
(2) $\forall x \in S$: $B \longmapsto p(x, B)$ is prob. measure on (S, Σ)

Def
of

Thm
on (
Then
r.v.'s

and
$\forall$

for $\mathcal{F}$

Def A Markov chain is **time-homogeneous** if transition probabilities do not depend on n.

Thm Let $\{p_n\}_{n\geq 0}$ be transition probabilities on (S,Σ). Let μ = prob. measure on (S,Σ).
Then $\exists\ (\Omega,\mathcal{F},P)$ = prob. space supporting r.v.'s $\{X_n\}_{n\geq 0}$ s.t.

$$\forall B\in\Sigma:\quad P(X_0\in B)=\mu(B)$$

and

$$\forall n\geq 0\ \forall B\in\Sigma:\quad P(X_{n+1}\in B\mid \mathcal{F}_n)=p_n(X_n,B) \qquad (*)$$

for $\mathcal{F}_n := \sigma(X_0,\dots,X_n)$.

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$S, \Sigma)$.

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$p_n(X_n, B)$

(*)

Pf: Let $(\Omega, \mathcal{F}, P)$ be prob. spaces supporting independent r.v.'s $\{\xi(x,n) : \begin{smallmatrix} x \in S \\ n \geq 0 \end{smallmatrix}\} \cup \{X_0\}$

s.t. $P(\xi(x,n) \in B) = p_n(x, B) \quad (B \in \Sigma)$.

and $P(X_0 \in B) = \mu(B) \qquad (B \in \Sigma)$.

Define, recursively,

$$X_{n+1} := \xi_n(X_n, n)$$

Then for $\tilde{\mathcal{F}}_n := \sigma\left(X_0 \cup \{\xi(x,k) : x \in S, k < n\}\right)$

we have

$$P(X_{n+1} \in B \mid \tilde{\mathcal{F}}_n) = P\left(\xi_n(X_n, n) \in B \mid \tilde{\mathcal{F}}_n\right)$$

$$= p(X_n, B) \quad \text{a.s.}$$

RHS is $\mathcal{F}_n$-meas, so we get (*). $\boxtimes$

Examples: $p(x,\{y\}) =: p(x,y)$

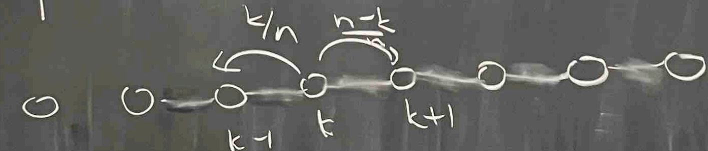
- **Two-state chain** $S = \{0,1\}$.

stochastic matrix

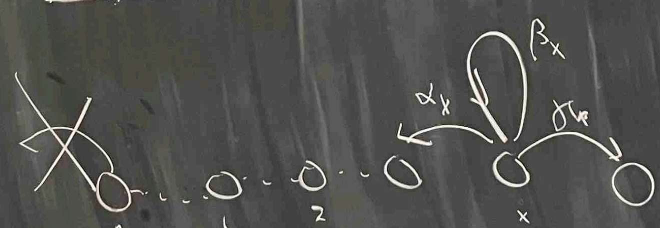
$$p = \begin{pmatrix} 2/5 & 3/5 \\ 1/4 & 3/4 \end{pmatrix}$$

- **Ehrenfest urn** $S = \{0,1,\ldots,n\}$

$$p(k,k-1) := \frac{k}{n}, \quad p(k,k+1) := \frac{n-k}{n}$$



- **Birth-death chain** $S = \mathbb{N} = \{0,1,\ldots\}$



$$p(x,x-1) = \alpha_x$$
$$p(x,x+1) = \gamma_x$$
$$p(x,x) = \beta_x$$

$\alpha_x, \beta_x, \gamma_x \geq 0, \quad \alpha_x + \beta_x + \gamma_x = 1$

$\alpha_0 = 0.$

- card s

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split

- random

$\{Z_k\}_{k\geq 0}$

Defin

p

E.g.

- **card shuffle** $\quad S = \mathcal{S}_n = \{\text{permutations of } \{1, \dots, n\}\}$

g = prob. mass function on S.

$$p(\sigma, \tilde{\sigma}) = g(\sigma^{-1} \circ \tilde{\sigma})$$

then the chain is a "random walk" on $\mathcal{S}_n$.

split the deck shuffle:

$$g(\sigma) = \frac{1}{n-1} \text{ on } \{(k+1, \dots, n, 1, \dots, k) : k = 1, \dots, n-1\}$$

- **random walk** (proper) $\quad S = G$ = additive group

$\{Z_k\}_{k \geq 0}$ iid $\sim \nu$ = prob. measure on G.

Define $X_n = Z_1 + \dots + Z_n$.

$$p(x, B) = \nu(-x + B) = \nu(\{-x + y : y \in B\})$$

e.g. $G = \mathbb{Z}^d, \mathbb{R}^d$

mutations $\}$
of $\{1, \ldots, n\}$ $\}$

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k" on $\mathcal{S}_n$.

$\ldots, k)$: $k = 1, \ldots, n-1\}$

= additive group

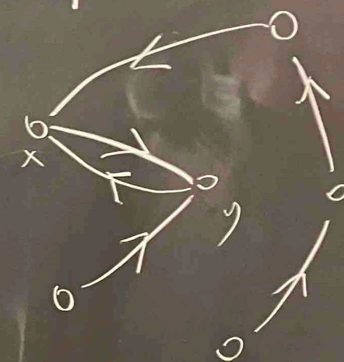
on G.

$-x+y : y \in B\}$)

- random walk on graphs

Let V = discrete set, $p : V \times V \to [0,1]$ prob. tr. kernel

<u>Define</u> $E = \{(x,y) \in V \times V : p(x,y) > 0\}$.



encodes possible transitions.

<u>Reverse this:</u>

Weighted graph: (V, E, w)

$w : E \to \mathbb{R}_+$ weight.

then $p(x,y) := \dfrac{w(x,y)}{\sum_{z : (x,z) \in E} w(x,z)}$ is transition kernel (if denominator > 0)

- <u>simple exclusion</u>

walkers on $G=(V,E)$, restricted to at most one individual at each vertex.

$$S=\{0,1\}^V \ni \eta,\tilde{\eta}$$

tr. prob. for one walker

$$\mathbb{P}(\eta,\tilde{\eta})=\begin{cases} \frac{1}{|V|}\,p(x,y) & \tilde{\eta}_z=\eta_z \quad z\neq x,y \\ & (\eta_x,\eta_y)=(1,0) \\ & (\tilde{\eta}_x,\tilde{\eta}_y)=(0,1) \\ 0 & \text{all other } \eta\neq\tilde{\eta} \text{ cases.} \\ \text{rest} & \text{for } \eta=\tilde{\eta} \end{cases}$$

interacting particle system