<u>Thm</u> (Subadditive Ergodic Thm) Let $\{X_{m,n} : 0 \le m < n\}$ be r.v. st

(1) $\forall\, 0 \le m < n$: $X_{0,n} \le X_{0,m} + X_{m,n}$

(2) $\forall k \ge 0$: $\{X_{nk,\, n(k+1)} : n \ge 1\}$ stationary

(3) $\forall m \ge 0$: $\{X_{m,\, m+n} : n \ge 1\} \overset{\text{law}}{=} \{X_{0,n} : n \ge 1\}$,

(4) $EX_{0,1}^{+} < \infty \;\wedge\; \exists \gamma_0 > -\infty,\ \forall n \ge 0: EX_{0,n} \ge -\gamma_0 n$

Then

(a) $\gamma := \lim_{n\to\infty} \frac{EX_{0,n}}{n} \in \mathbb{R}$ exists

(b) $X := \lim_{n\to\infty} \frac{X_{0,n}}{n}$ exists (in $\mathbb{R}$) a.s., and in L^1, $X \in L^1$

(c) if (2) is ergodic $\Rightarrow$ $X = \gamma$ a.s.

last time:

$$\overline{X} := \limsup_{n\to\infty} \frac{X_{0,n}}{n}$$

$$\underline{X} := \liminf_{n\to\infty} \frac{X_{0,n}}{n}$$

Lemmas:

$$E\overline{X} \le \gamma$$

$$E\underline{X} \ge \gamma$$

These imply $\underline{X} = \overline{X}$ a.s.

NTS
- limits in L^1
- if ergodic, then $X = \gamma$ a.s.

$$(X_{0,n} - nX)_+ \le \sum_{l=0}^{n-1} (X_{l,l+1} - X)_+$$

$$\left(\frac{X_{0,n}}{n} - X\right)_+ \le \frac{1}{n} \sum_{l=0}^{n-1} \underbrace{(X_{l,l+1} - X^{(l)})_+}_{\text{stationary}}$$

RHS converges in L^1 by Birkhoff's Thm.

So $\left\{ \left(\frac{X_{0,n}}{n} - X\right)_+ : n \ge 1 \right\}$ is UI.

Now

$$E\left(\left|\frac{X_{0,n}}{n} - X\right|\right) = 2E\left(\left(\frac{X_{0,n}}{n} - X\right)_+\right) - E\left(\frac{X_{0,n}}{n} - X\right)$$

RHS tends to zero because:

$$E\left(\left(\frac{X_{0,n}}{n} - X\right)_+\right) \longrightarrow 0 \text{ by UI \& a.s. conv.}$$

$$E\left(\frac{X_{0,n}}{n}\right) \longrightarrow \gamma = EX$$

If seq. in (2) is ergodic for each $k \geq 1$.

then we proved $X \leq \gamma$ a.s.

Since $EX = \gamma \implies X = \gamma$ a.s. $\boxtimes$

Examples: $\{S_n\}$ = random walk on $\mathbb{Z}^d$

$S_n = Z_1 + \cdots + Z_n$, $\{Z_i\}$ iid, $\mathbb{Z}^d$-valued,

$R_{l,n} = |\{S_l, \ldots, S_{n-1}\}|$

$R_{0,n} \leq R_{0,m} + R_{m,n}$ (1)

$R_{kn,(k+1)n} = |\{S_{kn}, \ldots, S_{(k+1)n-1}\}| = |\{0, \ldots, S_{(k+1)n-1} - S_{kn}\}| \in \sigma(X_{kn}, \ldots, X_{(k+1)n-1})$

independent for different k $\implies$ (2)

(3) similar

(4) automatic: $0 \leq R_{0,n} \leq n$.

So This gives: $\lim_{n\to\infty} \frac{R_{0,n}}{n}$ exists a.s.

In fact: limit equals $P(\forall k \geq 1: S_k \neq 0)$

Longest common subsequence $\{Y_n\}, \{Z_n\}$ random variables

$$X_{mn} := \max\left\{ k = 0, \dots, n-m : \begin{pmatrix} \exists\, m \leq i_1 < \dots < i_k \leq n \\ \exists\, m \leq j_1 < \dots < j_k \leq n \end{pmatrix} : \forall l = 1 \dots k : Y_{i_l} = Z_{j_l} \right\}$$

Then $X_{0n} \geq X_{0,m} + X_{m,n}$

If e.g. Y, Z are iid, then $\lim_{n\to\infty} \frac{X_{0,n}}{n}$ exists a.s. & is constant.

First-passage percolation

$$\rho(x,y) = \inf \Big\{ \sum_{i=1}^{n} \underbrace{L_{x_{i-1},x_i}}_{\text{passage time of edge } (x_{i-1},x_i)} : \{(x_{i-1},x_i)\}_{i=1}^{n} = \text{path for } x \to y \Big\}$$

Thm Assuming $\{L_{u,v} : (u,v) \in E(\mathbb{Z}^d)\}$ stationary w.r.t. translates and $E L_{u,v} < \infty$ then:

$$\forall x \in \mathbb{Z}^d, \quad d(x) := \lim_{n\to\infty} \frac{\rho(0,nx)}{n} \text{ exist a.s. \& } L^1$$

Moreover, if $\{L_{u,v} : (u,v) \in E(\mathbb{Z}^d)\}$ is ergodic, then $d(\cdot)$ is non-random.

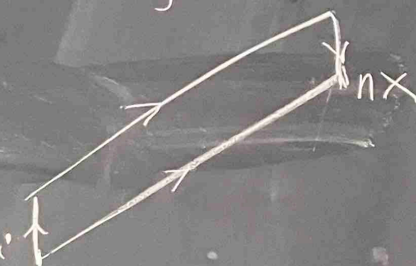
Pf: Verify conditions of SET

Will only prove last part.

Use subadditivity again: Let $e_i = (0,\dots,0,\underset{\uparrow\, i\text{th-coord}}{1},0\dots0)$

Then

$$\rho(0,nx) \le \rho(0,e_i) + \rho(e_i, e_i+nx) + \rho(nx, e_i+nx)$$



Lemma $\{Z_n\}$ stationary, $Z_0 \in L^1$

$$\Rightarrow \lim_{n\to\infty} \frac{Z_n}{n} = 0 \quad \text{a.s.}$$

Pf: $$\sum_{n=0}^{\infty} P(|Z_n| > \varepsilon n) = \sum_{n=0}^{\infty} P(|Z_0| > \varepsilon n)$$
$$\le \frac{1}{\varepsilon} E|Z_0| \qquad \boxtimes$$

Borel-Cantelli implies the claim.

$\rho(nx, nx+e_i) \le L_{nx, nx+e_i}$ ... stationary & $\in L^1$

$$d_\omega(x) \le d_{\theta_{e_i}(\omega)}(x) \quad \text{a.s.}$$

environment

Applying $-e_i$-shift $\quad d_{\theta_{e_i}(\omega)}(x) \le d_\omega(x)$

If law $\{L_{xy} : (x,y) \in E(\mathbb{Z}^d)\}$ jointly ergodic wrt shift $\Rightarrow$ d invariant functions and so constant a.s. $\boxtimes$

Fact $x \mapsto d(x)$ extends to a norm on $\mathbb{R}^d$.

Implies shape theorem for balls in this metric.

Thm Assuming ergodicity of $\{L_{xy} : (x,y)\in E(\mathbb{Z}^d)\}$ and $L_{xy}\in L^1$, we get: $\forall\, \varepsilon>0$,

$$(1-\varepsilon)\{x\in\mathbb{R}^d : d(x)\le n\}\cap\mathbb{Z}^d$$

$$\{x\in\mathbb{Z}^d : \rho(0,x)\le n\}$$

$$\subseteq (1+\varepsilon)\{x\in\mathbb{R}^d : d(x)\le n\}$$

with probability tending to 1 as $n\to\infty$.