Subadditivity

$\{a_n\}_{n\geq 1}$: $\forall m,n \geq 1$: $a_{n+m} \leq a_n + a_m$

$\Rightarrow \lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n} \in \mathbb{R} \cup \{-\infty\}$.

<u>Corollary</u>: $\forall n \geq 1$: $a_n \geq cn$ where $c := \lim_{n\to\infty} \frac{a_n}{n}$.

<u>Ex</u> Counting self-avoiding paths on transitive fin. degree graph.

$c_n = \#$ s.a. paths of length n started at some $o \in G$.

$\Rightarrow c_{n+m} \leq c_n c_m \Rightarrow \lim_{n\to\infty} c_n^{1/n}$ exists

<u>Ex</u> $T : B \to B$ bdd linear operator, B = Banach space

$\lim_{n\to\infty} \|T^n\|^{1/n}$ exists $\quad \|T^{n+m}\| \leq \|T^n\| \|T^m\|$

Ex $\{S_n\}_{n\geq 0}$ = Random walk on $\mathbb{Z}^d$

$S_n = X_1 + \cdots + X_n$, $\{X_i\}_{i\geq 0}$ iid, $\mathbb{Z}^d$-val'd.

$R_n := |\{S_0, \ldots, S_{n-1}\}|$ range of RW.

Since:

$$|\{S_0, \ldots, S_{n+m-1}\}| \leq |\{S_0, \ldots, S_{n-1}\}| + |\{S_n, \ldots, S_{n+m-1}\}|$$

we get

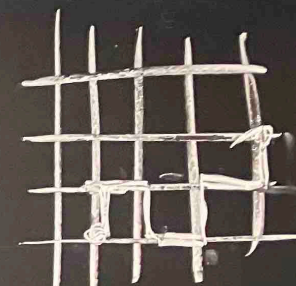
$$R_{n+m} \leq R_n + R_m', \quad R_m' \overset{d}{=} R_m$$

So

$$\mathbb{E}R_{n+m} \leq \mathbb{E}R_n + \mathbb{E}R_m$$

and

$$\lim_{n\to\infty} \frac{\mathbb{E}R_n}{n} \text{ exists.}$$



Q: Does $\lim_{n \to \infty} \frac{R_n}{n}$ exist?

First-passage percolation. Assign a positive random number L_{xy} to every (undirected) edge of $\mathbb{Z}^d$.

E.g. i.i.d, or stationary w.r.t. shifts.

Define $\forall x, y \in \mathbb{Z}^d$:

$$\rho(x,y) = \inf\left\{ \sum_{i=1}^{n} L_{x_{i-1},x_i} : \begin{array}{l} x_0 = x, x_1, \dots, x_n = y, n \geq 1 \\ \forall i = 1 \dots n;\ (x_{i-1}, x_i) \text{ are neighbors} \end{array} \right\}$$

Then ρ is a random metric on $\mathbb{Z}^d$.

Q: large scale behavior, specifically does

$$\lim_{n \to \infty} \frac{\rho(0, \lfloor nx \rfloor)}{n} \text{ exist } \forall x \in \mathbb{Z}^d \text{ ?}$$

Note $\rho(0, (n+m)x) \leq \rho(0, nx) + \rho(nx, (n+m)x)$

Thm (Subadditive Ergodic Thm; Kingman '68, Liggett '85)

Let $\{X_{m,n}\}_{0<m<n}$ are r.v's s.t.:

(1) $\forall\, 0<m<n$: $X_{0,n} \le X_{0,m} + X_{m,n}$ a.s.

(2) $\forall\, k \ge 1$: $\{X_{nk,(n+1)k}\}_{n\ge 1}$ is stationary

(3) $\forall\, m \ge 0$: $\{X_{m,m+n}\}_{n\ge 1} \overset{\text{law}}{=} \{X_{0,n}\}_{n \ge 1}$

(4) $E X_{0,1}^{+} < \infty \ \wedge\ \exists\, \gamma_0 > -\infty\ \forall n \ge 1$: $E X_{0,n} \ge \gamma_0 n$.

Then

(a) $\gamma := \lim_{n\to\infty} \frac{EX_{0,n}}{n} \in \mathbb{R}$ exists and $\gamma := \inf_{n\ge 1} \frac{EX_{0,n}}{n}$

(b) $X := \lim_{n\to\infty} \frac{X_{0,n}}{n}$ exists a.s. & L^1 and $X \in L^1 \wedge EX = \gamma$.

(c) If sequence in (2) ergodic $\forall k \ge 1$, then $X = \gamma$ a.s.

Proof of (a) (1) $\Rightarrow$ $X_{0,n} \le \sum_{\ell=1}^{n} X_{\ell-1,\ell} \le \sum_{\ell=1}^{n} X^{+}_{\ell-1,\ell}$

(2) $\Rightarrow$ $E X_{0,n} \le n E X^{+}_{0,1}$

(4) $\Rightarrow$ $|E X_{0,n}| \le \tilde{c} n$

Since $E X_{0,n+m} \overset{(1)}{\le} E X_{0,n} + E X_{m,m+n}$

$\overset{(2)}{=} E X_{0,n} + E X_{0,m}$

Fekete $\Rightarrow \gamma = \lim_{n \to \infty} \frac{E X_{0,n}}{n}$ exists, $|\gamma| \le \tilde{c}$.

Def $\overline{X}^{(m)} := \limsup_{n \to \infty} \frac{X_{m,m+n}}{n}$

$\underline{X}^{(m)} := \liminf_{n \to \infty} \frac{X_{m,m+n}}{n}$

Lemma $\forall m \ge 1$: $\overline{X}^{(m)} = \overline{X}^{(0)} \wedge \underline{X}^{(m)} = \underline{X}^{(0)}$ a.s.

Pf $\quad \dfrac{X_{0,n}}{n} \le \dfrac{X_{0,m}}{n} + \dfrac{n-m}{n}\,\dfrac{X_{m,n-m}}{n-m}$

Taking $n \to \infty$, we get

$$\overline{X}^{(0)} \le \overline{X}^{(m)}, \quad \underline{X}^{(0)} \le \underline{X}^{(m)}$$

Since (3) $\Rightarrow \overline{X}^{(0)} \overset{\text{law}}{=} \overline{X}^{(m)} \wedge \underline{X}^{(0)} \overset{\text{law}}{=} \underline{X}^{(m)}$

$\Rightarrow$ a.s. equality holds. $\boxtimes$

$$\boxed{\text{Denote } \overline{X} := \overline{X}^{(0)}, \ \underline{X} := \underline{X}^{(0)}.}$$

Lemma $\overline{X}^{+} \in L^1 \wedge E\overline{X} \le \gamma$.

if sequence in (2) is ergodic $\forall k \ge 1$

then $\overline{X} \le \gamma$ a.s.

Pf Fix $k \geq 1$, write $n = mk + r$, $r = 0, \dots, k-1$.

$$X_{0,n} \leq \sum_{j=0}^{m-1} X_{jk,(j+1)k} + X_{mk,n}$$

Then
$$\frac{X_{0,n}}{n} \leq \frac{m}{mk+r} \frac{1}{m} \sum_{j=0}^{m-1} X_{jk,(j+1)k} + \frac{\sum_{\ell=km}^{km+k-1} |X_{\ell,\ell+1}|}{n}$$

Fact $\{Z_n\}$ stationary, $Z_1 \in L^1$: $\lim_{n\to\infty} \frac{|Z_n|}{n} = 0$ a.s.

Birkhoff $\frac{1}{m} \sum_{j=0}^{m-1} X_{jk,(j+1)k} \longrightarrow Y_k$ a.s. & L^1.

So $\overline{X} \leq \frac{Y_k}{k}$ a.s. $(\forall k \geq 1)$ $\Rightarrow \overline{X}^+ \in L^1$

Know $EY_k = E X_{0,k}$

and $E\overline{X} \leq \lim_{k\to\infty} \frac{1}{k} E X_{0,k} = \gamma$. $\boxtimes$

Lemma $\underline{X}^{-} \in L^{1} \wedge E\underline{X} \geq \gamma$

Pf Fix $\varepsilon > 0$, $M > 0$, $N \geq 1$.

Define $Y_{m,n} := X_{m,n} - [\underline{X} \vee (-M)](n-m)$

Note $\liminf_{n \to \infty} \frac{Y_{m,n}}{n} = \underline{X} - [\underline{X} \vee (-M)] \leq 0$.

So $\forall m \geq 1$: $T_m := \inf\{n \geq 1 : Y_{m,m+n} < \varepsilon n\} < \infty$ a.s.

Define $\{\tau_k\}_{k \geq 0}$ by $\tau_0 := 0$ and

$$\forall k \geq 0: \quad \tau_{k+1} := \begin{cases} T_{\tau_k} & T_{\tau_k} \leq N \\ 1 & T_{\tau_k} > N \end{cases}$$

For $n \geq 1$ set $K_n := \max\{k \geq 0 : \tau_k \leq n\}$.

Then $$Y_{0,n} \leq \sum_{k=1}^{K_n} Y_{\tau_{k-1},\tau_k} + Y_{\tau_{K_n},n}$$

Observe: $$Y_{\tau_{k-1},\tau_k} \leq \varepsilon(\tau_k - \tau_{k-1})1_{\{T_{\tau_{k-1}} \leq N\}} + Y_{\tau_{k-1},\tau_{k-1}+1}1_{\{T_{\tau_{k-1}} > N\}}$$

to get $$Y_{0,n} \leq \varepsilon n + \sum_{\ell=0}^{n-1} Y_{\ell,\ell+1} 1_{\{T_\ell > N\}} + \sum_{j=0}^{N} |Y_{n-j,n}|$$

Take exp $$\frac{1}{n} E Y_{0,n} \leq \varepsilon + E(Y_{0,1} 1_{T_0 > N}) + \frac{1}{n}\sum_{j=0}^{N} E|Y_{0,j}|$$

So taking $n \to \infty$, $N \to \infty$, $\varepsilon \downarrow 0$ we get $E(\underline{X} \vee (-M)) \geq \gamma$.

We get $E\underline{X} \geq \gamma$ by MCT. $\boxtimes$