MATH 275C Spring 2025

Final Exam

Due: Thr June 19, 11:59 PM PDT

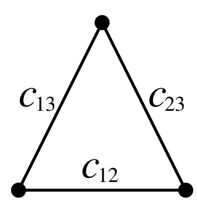
Instructions: There are 5 problems. Solve each of these on your own on a separate page. You may draw on the theorems proved in class but you have to make the statement of the relevant theorem appear explicitly in your solution. Then typeset your solutions in TeX and send them, by the above due date/time, as one PDF file to biskup@math.ucla.edu. Work on your own; internet search is allowed solely for the purpose of checking the definitions of basic concepts.

Problem 1: Let $\mathscr{X}$ be a compact metric space endowed with the σ-algebra $\mathcal{G} := \mathcal{B}(\mathscr{X})$ of its Borel sets. Suppose that $\varphi\colon \mathscr{X} \to \mathscr{X}$ is a continuous map. Prove that there exists a probability measure μ such that $(\mathscr{X}, \mathcal{G}, \mu, \varphi)$ is a measure-preserving system.

Problem 2: Let S be a countable set, P a Markov transition kernel on S and μ a stationary probability measure for P. Endow $\mathscr{X} := S^{\mathbb{N}}$ with the σ-algebra $\mathcal{G} := \mathcal{P}(S)^{\otimes\mathbb{N}}$, where $\mathcal{P}(S)$ is the powerset of S, and let $\theta\colon \mathscr{X} \to \mathscr{X}$ denote the left shift. Define P^μ be the law of the Markov chain with initial distribution μ. Do as follows:

(1) Prove that $(\mathscr{X}, \mathcal{G}, P^\mu, \theta)$ is a measure-preserving system, called the *Markov shift*.
(2) Assuming that the Markov chain is irreducible, prove that P^μ is ergodic.

Problem 3: Consider an electric network on the triangle graph with conductance between vertices i and j denoted as c_{ij}; see the picture:



Let $C_{\mathrm{eff}}(i,j)$ denote the effective conductances between vertices i and j. Compute $C_{\mathrm{eff}}(1,2)$ as a function of the $c_{i,j}$'s. Work preferably directly from the definition $C_{\mathrm{eff}}(1,2)$. Give proofs of all "network reduction" arguments/tools you use.

Problem 4: Consider the Q-matrix on $S := \{0,1,\dots\}$ defined by

$$q(x, x+1) := \alpha x^r$$
$$q(x, x-1) := \beta x^r$$
$$q(x, x) := -(\alpha+\beta) x^r$$

Identify the values $\alpha, \beta, r > 0$ for which the minimal solution to Backward Kolmogorov Equations is stochastic (and the resulting Markov chain is thus non-explosive).

Problem 5: Let $\{X_t\}_{t\geqslant 0}$ be a continuous-time Markov chain on a countable state space S. Given a function $\sigma\colon S \to (0,\infty)$ with $0 < \inf\sigma \leqslant \sup\sigma < \infty$, let

$$\tau(t) := \inf\Big\{u \geqslant 0\colon \int_0^u \sigma(X_s)\,\mathrm{d}s \geqslant t\Big\}$$

and set

$$Y_t := X_{\tau(t)}.$$

Prove that $\{Y_t\}_{t\geqslant 0}$ is a continuous time Markov chain on S and identify the Q-matrix of Y in terms of that of X. (The process Y is a *time-change* of X by *speed measure* σ.)