28. SUBADDITIVE ERGODIC THEOREM

Our last item of our business in this section is a very useful version of the Pointwise Ergodic Theorem which appears in a number of probability contexts. The key difference is that instead of sums of "translates" of a function by a m.p.t., we are dealing with subadditive families of random variables.

28.1 Statement and examples.

Throughout probability and statistical mechanics, subadditivity serves a useful "soft" tool for proving limits which would otherwise seem impossible. The observation that this is the case goes back to a lemma usually attributed to M. Fekete, although one would have hard time finding this statement in the corresponding paper:

Lemma 28.1 (Subadditive convergence; Fekete 1923) *Let $\{a_n\}_{n\geqslant 1}$ be an $\mathbb{R}$-valued sequence which is subadditive in the sense*

$$\forall m, n \geqslant 1: \quad a_{n+m} \leqslant a_n + a_m \tag{28.1}$$

Then

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geqslant 1} \frac{a_n}{n} \in \mathbb{R} \cup \{-\infty\} \tag{28.2}$$

Proof. Denote the infimum as c and suppose first that $c > -\infty$. Given $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that $a_m/m \leqslant c + \epsilon$. Any $n \in \mathbb{N}$ can then be written as $n = km + r$ for $k \geqslant 0$ and $r \in \{0, 1, \dots, m-1\}$. Subbaditivity then implies

$$a_{km+r} \leqslant k a_m + a_r \tag{28.3}$$

which yields

$$\frac{a_{km+r}}{mk+r} \leqslant \frac{km}{km+r}\frac{a_m}{m} + \frac{1}{km+r} \max_{0\leqslant j<m} a_j \tag{28.4}$$

Invoking the defining property of m and returning to the notation using n we get

$$c \leqslant \frac{a_n}{n} \leqslant c + \epsilon + \frac{1}{n}\Bigl(\max_{0\leqslant j<m} a_j + m|c+\epsilon|\Bigr) \tag{28.5}$$

for all $n \geqslant 1$. Taking $n \to \infty$ followed by $\epsilon \downarrow 0$ we get (28.2).

For $c = -\infty$ we instead pick $M > 0$ and find m so that $a_m/m \leqslant -M$. Then we repeat the same argument to show that

$$\limsup_{n\to\infty} \frac{a_n}{n} \leqslant -M \tag{28.6}$$

Taking $M \to \infty$ then yields the claim. □

In 1968, J. Kingman considered a random generalization of a subadditive sequence in the form of an array $\{X_{m,n}\}_{0\leqslant m<n}$ satisfying

$$\forall \ell < m < n: \quad X_{\ell,n} \leqslant X_{\ell,m} + X_{m,n} \tag{28.7}$$

Here we think of $X_{m,n}$ as a quantity assigned to the interval $[m,n)$. Under additional stationarity assumptions,J. Kingman then proved existence of the a.s.-limit $\lim_{n\to\infty} n^{-1} X_{0,n}$. Later H. Kesten introduced a generalization, which he referred to as superconvolutive sequence, which relaxed some of Kingman's conditions. In mid 1980s T. Liggett found another extension (which was immediately applied by R. Durrett in his study of the contact process) that turns out to be more applicable than its predecessors.

Theorem 28.2 (Liggett's Subadditive Ergodic Theorem, 1985) *Consider a triangular array* $\{X_{m,n}\}_{0\leqslant m\leqslant n}$ *of random variables such that*

(1) $\forall 0 < m < n\colon\ X_{0,n} \leqslant X_{0,m} + X_{m,n}$ *a.s.*
(2) $\{X_{nk,n(k+1)}\}_{n\geqslant 1}$ *is stationary for all* $k \geqslant 1$.
(3) *The law of* $\{X_{m,m+n}\}_{n\geqslant 1}$ *does not depend on* m.
(4) $\mathbb{E}X_{0,1}^{+} < \infty$ *and there exists* $\gamma_0 > -\infty$ *such that* $\forall n \geqslant 1\colon\ \mathbb{E}X_{0,n} \geqslant \gamma_0 n$.

Then the following holds:

(a) *The limit* $\gamma := \lim_{n\to\infty} \frac{\mathbb{E}X_{0,n}}{n}$ *exists, is finite and equals* $\inf_{n\geqslant 1} \frac{\mathbb{E}X_{0,n}}{n}$.
(b) $X := \lim_{n\to\infty} \frac{X_{0,n}}{n}$ *exists a.s. and in* L^1 *with* $EX = \gamma$.
(c) *If the sequence in (2) above is ergodic for all* $k \geqslant 1$*, then* $X = \gamma$ *a.s.*

Before we delve into the proof, let us check some examples of sequences satisfying properties (1-4) of this theorem:

***Example* 28.3** (Additive functionals) Given a stationary sequence $\{Z_n\}_{n\geqslant 0}$ with $Z_0 \in L^1$, let $X_{m,n} = \sum_{j=m}^{n-1} Z_j$. Then properties (1-4) hold for $\{X_{m,n}\}_{0<m<n}$. The conclusion of Theorem 28.2 is then the Pointwise Ergodic Theorem. (Nothing new is learned though since the proof of Theorem 28.2 uses the Pointwise Ergodic Theorem.)

***Example* 28.4** (Range of a random walk) Let $\{S_n\}_{n\geqslant 0}$ denote a random walk on $\mathbb{R}^d$ (i.e., $S_n = Z_1 + \cdots + Z_n$ for $\{Z_i\}_{i\geqslant 1}$ i.i.d. $\mathbb{R}^d$-valued random variables) and write

$$X_{m,n} := \big|\{S_m, \dots, S_{n-1}\}\big| \tag{28.8}$$

for the number of sites visited by the walk in time interval $[m, n)$. This set $X_{0,n}$ is sometimes called the *range* of the walk. Note that for $\ell < m < n$ the union bound gives

$$\big|\{S_\ell, \dots, S_{n-1}\}\big| \leqslant \big|\{S_\ell, \dots, S_{m-1}\}\big| + \big|\{S_m, \dots, S_{n-1}\}\big| \tag{28.9}$$

the array $\{X_{m,n}\}_{0<m<n}$ satisfies condition (1) of Theorem 28.2. The other conditions are checked using the i.i.d. nature of $\{Z_i\}_{i\geqslant 1}$ — in fact, it suffices that $\{Z_i\}_{i\geqslant 1}$ is stationary.

***Example* 28.5** (Longest common subsequence) Consider stationary processes $\{Y_n\}_{n\geqslant 0}$ and $\{Z_n\}_{n\geqslant 0}$ and let $X_{m,n}$ denote the longest common subsequence for indices in interval $[m, n)$ defined as

$$X_{m,n} = \max\left\{k \geqslant 0\colon\ \begin{array}{l} \exists m < i_1 < \cdots < i_k \leqslant n,\ \exists m \leqslant j_1 < \cdots < j_k < n \\ \text{such that } X_{i_\ell} = Y_{j_\ell} \text{ for all } \ell = 1, \dots, k \end{array}\right\} \tag{28.10}$$

Then $X_{0,n} \geqslant X_{0,m} + X_{m,n}$ and so $\{-X_{m,n}\}_{0<m<n}$ satisfies condition (1) above. The other conditions are verified using stationarity of $\{Y_n\}_{n\geqslant 0}$ and $\{Z_n\}_{n\geqslant 0}$.

***Example* 28.6** (Distance in First-Passage Percolation) Consider the graph with vertex set $\mathbb{Z}^d$ and an undirected edge between every pair $x, y \in \mathbb{Z}^d$ with $\|x - y\|_1 = 1$. Writing $E(\mathbb{Z}^d)$ for the set of edges, suppose $\{W(e)\colon e \in E(\mathbb{Z}^d)\}$ are i.i.d. positive random variables with $W(e)$ interpreted as the length of (or, alternatively, time it takes to cross) edge e. For each distinct $x, y \in \mathbb{Z}^d$ define

$$D(x,y) := \inf\left\{ \sum_{j=1}^{n} W(x_{j-1}, x_j) \colon \begin{array}{l} \{(x_{i-1}, x_i) \colon i = 1, \dots, n\} \subseteq E(\mathbb{Z}^d) \\ x_0 = x,\, x_n = y, n \geqslant 1 \end{array} \right\} \tag{28.11}$$

to be the minimal length of any path from x to y, where the length of the path is simply the sum of the the W's for all edges in the path. In the alternative description using time to cross the edge, $D(x,y)$ is the first time once can get from x to y (or y to x). This version of the problem (introduced originally by J. Hammersley and D. Welsh in 1965) earned it the name *First-Passage Percolation*.

For this setup, given any $x \in \mathbb{Z}^d$ and naturals $0 \leqslant m < n$, let

$$X_{m,n} := D\big(mx, (n-1)x\big) \tag{28.12}$$

The triangle inequality obeyed by D then implies condition (1) in Theorem 28.2 and conditions (2-3) then hold by the stationarity of the weights W. For condition (4) we need to assume $W \in L^1$.

28.2 Proof of Subadditive Ergodic Theorem.

We will now move to the proof of Theorem 28.2. Our first item of business is the convergence under expectation:

Proof of Theorem 28.2(a). First let us check integrability of all random variables. Condition (1) directly translates into $X_{m,n}^+ \leqslant X_{0,m}^+ + X_{m,n}^+$. By iteration, $X_{0,n}^+ \leqslant \sum_{k=0}^{n-1} X_{k,k+1}^+$. By (4), this implies $EX_{0,n}^+ \leqslant nEX_{0,1}^+ < \infty$. Noting that $|X_{0,n}| = 2X_{0,n}^+ - X_{0,n}$ we get

$$E|X_{0,n}| \leqslant n(2EX_{0,1}^+ - \gamma_0) \tag{28.13}$$

where we again invoked (4). Conditions (1) and (3) in turn tell us

$$EX_{0,m+n} \leqslant EX_{0,m} + EX_{m,m+n} = EX_{0,m} + EX_{0,n} \tag{28.14}$$

The sequence $n \mapsto EX_{0,n}$ is subadditive and bounded by a linear function. Hence, by the Lemma 28.1, the limit in (a) exists finitely and equals the infimum. □

To study the a.s. limit, we introduce the random variables:

$$\overline{X}^{(m)} = \limsup_{n\to\infty} \frac{X_{m,m+n}}{n} \quad \text{and} \quad \underline{X}^{(m)} = \liminf_{n\to\infty} \frac{X_{m,m+n}}{n} \tag{28.15}$$

Our first goal is to show that these do not depend on m:

Lemma 28.7 *Under the conditions (1-4),*

$$\forall m \geqslant 1\colon \quad \overline{X}^{(m)} = \overline{X}^{(0)} \text{ a.s.} \quad \wedge \quad \underline{X}^{(m)} = \underline{X}^{(0)} \text{ a.s.} \tag{28.16}$$

Proof. Condition (1) implies

$$\frac{X_{0,n}}{n} \leqslant \frac{X_{0,m}}{n} + \frac{n-m}{n}\frac{X_{m,m+(n-m)}}{n-m} \tag{28.17}$$

Taking $n \to \infty$ yields $\overline{X}^{(0)} \leqslant \overline{X}^{(m)}$ and $\underline{X}^{(0)} \leqslant \underline{X}^{(m)}$. But condition (3) gives $\overline{X}^{(0)} \overset{\text{law}}{=} \overline{X}^{(m)}$ and so we must have $\overline{X}^{(0)} = \overline{X}^{(m)}$ a.s. The infima are handled similarly. □

As a consequence, there is no need to distinguish between $\overline{X}^{(m)}$ and $\underline{X}^{(m)}$ for different m; from now on we will write simply $\overline{X}$, resp., $\underline{X}$ to denote any these random variables. In the course of the proof, we will need the following simple observations:

Lemma 28.8 *Let $\{Z_n\}_{n\geqslant 1}$ be stationary random variables with $Z_1 \in L^1$. Then*

$$\frac{Z_n}{n} \underset{n\to\infty}{\longrightarrow} 0 \quad \text{a.s.} \tag{28.18}$$

Proof. Using stationarity to get

$$\sum_{n\geqslant 1} P(|Z_n| > \epsilon n) = \sum_{n\geqslant 1} P(|Z_0| > \epsilon n) \leqslant \epsilon^{-1} E|Z_0| \tag{28.19}$$

this follows from the Borel-Cantelli lemma. □

The proof of Theorem 28.2 now proceeds by establishing bounds on $E\overline{X}$ and $E\underline{X}$:

Lemma 28.9 *Under the conditions (1-4), $(\overline{X})^+ \in L^1$ and $E\overline{X} \leqslant \gamma$. Moreover, if the sequence in (2) is ergodic for all $k \geqslant 1$, then $\overline{X} \leqslant \gamma$ a.s.*

Proof. Fix $m \geqslant 1$ and write each $n \geqslant 1$ as $n = mk + r$ with $0 \leqslant r < m$. Using condition (1) we then get

$$\frac{X_{0,n}}{n} \leqslant \frac{k}{km+r}\left[\frac{1}{k}\sum_{\ell=0}^{k-1} X_{\ell m,(\ell+1)m}\right] + \frac{X_{km,n}}{km+r} \tag{28.20}$$

By condition (2) and Birkhoff's Pointwise Ergodic Theorem, the bracket on the right-hand side tends to a random variable $Y_m \in L^1$ with the property $EY_m = EX_{0,m}$. For $n \geqslant m$ the last term is bounded by $\frac{1}{n}\sum_{j=1}^{m} |X_{n-j,n}|$ which tends to zero a.s. by Lemma 28.8, condition (2) and (28.13). Hence we get $\overline{X} \leqslant \frac{Y_m}{m}$ and thus $(\overline{X})^+ \in L^1$. Taking expectations we get $E\overline{X} \leqslant \frac{1}{m}EX_{0,m}$ which tends to γ as $m \to \infty$ by conclusion (a) of Theorem 28.2. To get the second part of the claim, note that under ergodicity assumption in (2), we have $Y_m = EX_{0,m}$ a.s. and the bound $\overline{X} \leqslant \gamma$ then holds pointwise a.s. □

Lemma 28.10 *Under the conditions (1-4), $(\underline{X})^- \in L^1$ and $E\underline{X} \geqslant \gamma$.*

Proof. We use an argument similar to that invoked in the proof of the Stephanov-Hopf Ratio Ergodic Theorem (Theorem 26.7). Pick $M > 0$ and consider the random variables

$$Y_{m,n} := X_{m,n} - (n-m)[\underline{X} \vee (-M)] \tag{28.21}$$

Note that, as the second term is additive, $Y_{m,n}$ obeys (1) if $X_{m,n}$ does. Since $(X_{m,n})^+ \in L^1$, we have $Y_{m,n} \in L^1$. Next, noting that $\liminf_{n\to\infty} Y_{m,m+n}/n = \underline{X} - [\underline{X} \vee (-M)] \leqslant 0$, for each $\epsilon > 0$ we have

$$T_n := \inf\{n \geqslant 1 \colon Y_{m,m+n} \leqslant \epsilon n\} < \infty \quad \text{a.s.} \tag{28.22}$$

In light of $Y_{0,1} \in L^1$, there is $N_0 \geqslant 1$ such that $E(Y_{0,1}1_{\{T_0>N\}}) < \epsilon$ holds for all $N \geqslant N_0$. Define the random variables

$$S_m := \begin{cases} T_m, & \text{if } T_m \leqslant N \\ 1, & \text{otherwise} \end{cases} \tag{28.23}$$

and

$$\tau_m := \tau_{m-1} + S_{\tau_{m-1}} \tag{28.24}$$

Note that $\tau_{m+1} - \tau_m \in \{1, \dots, N\}$. Now we use these random "times" in conjunction with (1) to write

$$Y_{0,n} \leqslant Y_{0,\tau_1} + Y_{\tau_1,\tau_2} + \dots + Y_{\tau_{K-1},\tau_K} + Y_{\tau_K,n} \tag{28.25}$$

where $K = \max\{k \colon \tau_k \leqslant n\}$. The various truncations invoked above now imply

$$Y_{\tau_\ell,\tau_{\ell+1}} \leqslant \epsilon(\tau_{\ell+1} - \tau_\ell)1_{\{T_{\tau_\ell} \leqslant N\}} + Y_{\tau_\ell,\tau_{\ell+1}}1_{\{T_{\tau_\ell}>N\}} \tag{28.26}$$

Summing the first terms over $\ell = 0, \dots, K-1$ yields a quantity less than ϵn, while the fact that $\tau_{\ell+1} = \tau_\ell + 1$ on $\{T_{\tau_\ell} > N\}$ bounds the sum of the second terms by that of $|Y_{\ell,\ell+1}|1_{\{T_\ell>N\}}$. Hence we get

$$Y_{0,n} \leqslant \epsilon n + \sum_{\ell=1}^{n-1} Y_{\ell,\ell+1}1_{\{T_\ell>N\}} + \sum_{j=0}^{N} |Y_{n-j,n}| \tag{28.27}$$

where we also observed that $n - \tau_K \leqslant N$. Taking expectations while invoking condition (2) yields

$$\frac{EY_{0,n}}{n} \leqslant \epsilon + E\big(|Y_{0,1}|1_{\{T_0>N\}}\big) + \frac{1}{n}\sum_{j=0}^{n} E|Y_{0,j}| \tag{28.28}$$

Taking $n \to \infty$, $N \to \infty$ and $\epsilon \downarrow 0$, the right-hand side tends to zero. The left-hand side in turn converges to $\gamma - E[\underline{X} \wedge (-M)]$. Hence $E(\underline{X} \wedge (-M)] \geqslant \gamma$. Taking $M \to \infty$ with the help of the Monotone Convergence Theorem we get thex claim. □

Now we can finish the remaining claims:

Proof of Theorem 28.2(b) and (c). Let us start with almost sure convergence: Since $\underline{X} \leqslant \overline{X}$ and, by Lemmas 28.9 and 28.10,

$$E\underline{X} \geqslant \gamma \geqslant E\overline{X} \tag{28.29}$$

we must have $\underline{X} = \overline{X}$ a.s. Under the ergodicity assumption in (2), we also get $\overline{X} \leqslant \gamma$ a.s. which in light of $E\overline{X} = \gamma$ forces $\overline{X} = \underline{X} = \gamma$ a.s.

Let X denote the a.s. limit. To prove convergence in L^1, we need to prove uniform integrability. For this we invoke subadditivity to get

$$(X_{0,n} - nX)^+ \leqslant \sum_{\ell=0}^{n-1}(X_{\ell,\ell+1} - X)^+ \overset{\text{a.s.}}{=} \sum_{\ell=0}^{n-1}(X_{\ell,\ell+1} - X^{(\ell)})^+ \tag{28.30}$$

where we also invoked Lemma 28.7. The variables under the sum on the right-hand side form a stationary sequence with $E(X_{\ell,\ell+1} - X^{(\ell)})^+ < \infty$ and so, by Birkhoff's Pointwise Ergodic Theorem the sum normalized by n converges a.s. and in L^1. It follows that

$\{(\frac{1}{n}X_{0,n}-X)^+\colon n\geqslant 1\}$ is UI and that $E(\frac{1}{n}X_{0,n}-X)^+\to 0$. The fact that $\frac{1}{n}EX_{0,n}\to\gamma=EX$ then proves $\frac{1}{n}X_{0,n}\to X$ in L^1 by way of the identity $|Z|=2Z^+-Z$. □

28.3 Applications.

Theorem 28.2 applies to Examples 28.3–28.6 and yields existence of an a.s. limit of the quantity $\frac{1}{n}X_{0,n}$ defined there. Let us comment further on two of the cases; namely, the range of the random walk in Example 28.3 and the First-Passage Percolation distance in Example 28.6. For the former we in fact get:

Lemma 28.11 *Let $\{S_n\}_{n\geqslant 1}$ be a random walk on $\mathbb{R}^d$ and set $R_n:=|\{S_1,\dots,S_n\}|$. Then*

$$\frac{R_n}{n}\underset{n\to\infty}{\longrightarrow}P\bigl(\forall k\geqslant 1\colon S_k\neq 0\bigr)\quad\text{a.s.}\tag{28.31}$$

Proof. Theorem 28.2 yields $R_n/n\to X$ a.s. and in L^1 with X constant a.s. by the Hewitt-Savage Zero-One Law. It thus suffices to compute the limit of expectations. Denote

$$T_0:=\inf\{k\geqslant 1\colon S_k=0\}\tag{28.32}$$

Then

$$\begin{aligned}P\bigl(S_{k+1}\notin\{S_1,\dots,S_k\}\bigr)&=P(X_k\neq 0,X_k+X_{k-1}\neq 0,\dots,X_k+\dots+X_1\neq 0)\\&=P(S_1,\dots,S_k\neq 0)=P(T_0>k)\end{aligned}\tag{28.33}$$

implies

$$ER_n=\sum_{k=0}^{n-1}P\bigl(S_{k+1}\notin\{S_1,\dots,S_k\}\bigr)=\sum_{k=0}^{n-1}P(T_0>k)\tag{28.34}$$

Since $P(T_0>k)\to P(T_0=\infty)$ as $k\to\infty$, the sum on the right normalized by n tends to $P(T_0=\infty)=P(\forall k\geqslant 1\colon S_k\neq 0)$ as well. □

We remark that a proof of (28.31) is possible that avoids any reference to the Subadditive Ergodic Theorem. We leave this to a homework exercise.

For Example 28.6 we in turn get:

Lemma 28.12 (Existence of FPP norm) *Suppose that the weights $\{W(e)\colon e\in E(\mathbb{Z}^d)\}$ in Example 28.6 are stationary with respect to the shifts of $\mathbb{Z}^d$. Assume that $EW(e)<\infty$ for all $e\in E(\mathbb{Z}^d)$. Then*

$$\forall x\in\mathbb{Z}^d\colon\ \rho(x):=\lim_{n\to\infty}\frac{1}{n}D(0,nx)\quad\text{exists a.s. and in }L^1\tag{28.35}$$

Moreover if $\{W(e)\colon e\in E(\mathbb{Z}^d)\}$ are ergodic in the sense that any $A\in\sigma(W(e)\colon e\in E(\mathbb{Z}^d))$ that is invariant under all shifts of $\mathbb{Z}^d$ has $P(A)\in\{0,1\}$, then $\rho(x)$ is constant a.s. for all $x\in\mathbb{Z}^d$.

Proof. A simple estimate using the path from 0 to x minimizing the number of edges used (and thus ℓ^1-distance) shows

$$0\leqslant ED(0,nx)\leqslant\|x\|_1\max_{e\in\mathbb{Z}^d}EW(e)\tag{28.36}$$

where the maximum is finite by our assumption that $W(e) \in L^1$ and stationarity. Theorem 28.2 then applies and yields (28.35). The main point is thus to prove a.s.-constancy of the limit under the assumption of joint ergodicity with respect to the shifts of $\mathbb{Z}^d$. This does not follow from clause (c) of Theorem 28.2 because joint ergodicity is weaker than ergodicity under the shifts by x only.

We have to show that that $\rho(x)$ is invariant under the shifts of the underlying "environment." Assume that the weights are realized as coordinate projections on the product space $\mathbb{R}_+^{E(\mathbb{Z}^d)}$; i.e., $W_\omega(x,y) := \omega(x,y)$ for $\omega \in \mathbb{R}_+^{E(\mathbb{Z}^d)}$ and $(x,y) \in E(\mathbb{Z}^d)$. Let τ_z be the map acting as

$$W_{\tau_z\omega}(x,y) = W_\omega(x+z, y+z) \tag{28.37}$$

and let $D_\omega(x,y)$ be the quantity from (28.11) and write $\rho_\omega(x)$ for the limit from (28.35). For each $z \in \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$, the triangle inequality gives

$$W_{\tau_z\omega}(0,nx) = W_\omega(z, z+nx) \leqslant W_\omega(z,0) + W_\omega(0,nx) + W_{\tau_{nx}\omega}(0,z) \tag{28.38}$$

Dividing by n, the first term on the right tends to zero as $n \to \infty$ while the second term tends to $\rho_\omega(x)$ a.s. For the third term we use Lemma 28.8 along with stationarity and finite mean to show convergence to zero a.s. It thus follows that $\rho_{\tau_z\omega}(x) \leqslant \rho_\omega(x)$ a.s. By symmetry, $\rho_{\tau_z\omega}(x) = \rho_\omega(x)$ a.s. implying that $\rho(x)$ is almost invariant under all the shifts of $\mathbb{Z}^d$. The assumed ergodicity then gives that $\rho(x) = E\rho(x)$ a.s. □

We remark that the existence of the limit along with $D(0,x) \overset{\text{law}}{=} D(-x,0)$ implies that $x \mapsto \rho(x)$ is positive homogeneous and symmetric under reflections through 0. Along with the triangle condition we then readily infer that ρ is (a restriction to $\mathbb{Z}^d$ of) a norm on $\mathbb{R}^d$. Using this norm we then show:

Theorem 28.13 (Shape theorem for First-Passage Percolation, J. Cox and R. Durrett, 1981) *Suppose $\{W(e)\colon e \in E(\mathbb{Z}^d)\}$ are ergodic with $W(e) \in L^1$ and let $B_r := \{x \in \mathbb{R}^d \colon \rho(x) \leqslant r\}$ be the ball of radius r in the associated (deterministic) norm ρ. Then*

$$\forall \epsilon \in (0,1)\colon\ P\biggl(B_{(1-\epsilon)r} \cap \mathbb{Z}^d \subseteq \bigl\{x \in \mathbb{Z}^d \colon D(0,x) \leqslant r\bigr\} \subseteq B_{(1+\epsilon)r}\biggr) \underset{r\to\infty}{\longrightarrow} 1 \tag{28.39}$$

The proof of this is quite easy when the weights W are bounded; indeed, one then only needs to control the growth of $n \mapsto D(0,nx)$ for a finite number of x and use the ℓ^1-distance to interpolate to get

$$\max_{\substack{x\in\mathbb{Z}^d \\ 0<\|x\|_\infty\leqslant n}} \frac{|D(0,x) - \rho(x)|}{n} \underset{r\to\infty}{\longrightarrow} 0 \tag{28.40}$$

The unbounded case is handled by truncating the distance to be less than M and showing that the limiting norm for the truncated weights tends to ρ as $M \to \infty$.