26. Recurrence in infinite measure spaces

We now move to discuss the differences that arise when the underlying measure space is infinite and/or the map we have is not measure preserving.

26.1 Halmos' Recurrence Theorem.

Motivated by the construction from the proof of the Poincaré Recurrence Theorem, we put forward:

Definition 26.1 (Wandering set) *Given a measurable map $\varphi\colon \mathscr{X} \to \mathscr{X}$ on a measurable space $(\mathscr{X}, \mathcal{G})$, a set $W \in \mathcal{G}$ is said to be wandering if*

$$W \cap \bigcup_{k\geqslant 1} \varphi^{-k}(W) = \varnothing \tag{26.1}$$

i.e., when $\forall x \in W\, \forall k \geqslant 1\colon \varphi^k(x) \notin W$.

An example of a wandering set is

$$W := A \smallsetminus \bigcup_{k\geqslant 1} \varphi^{-k}(A) \tag{26.2}$$

i.e., the set of $x \in A$ such that $\varphi^k(x) \notin A$ for all $k \geqslant 1$. Here are some additional elementary observations:

Lemma 26.2 *Let W be a wandering set. Then also $\varphi^{-1}(W)$ is wandering and every $W' \in \mathcal{G}$ with $W' \subseteq W$ is wandering. Moreover, we have*

$$\left\{\varphi^{-k}(W)\colon k \geqslant 0\right\} \text{ are disjoint} \tag{26.3}$$

In particular, if μ is a measure preserved by φ, then $\mu(\mathscr{X}) < \infty$ implies $\mu(W) = 0$.

Proof. That $\varphi^{-1}(W)$ and $W' \subseteq W$ are wandering as well as (26.3) follow directly from (26.1) and the properties of φ^{-1}. Since the sets in (26.3) are disjoint and of the same measure, $\mu(\mathscr{X}) < \infty$ forces $\mu(W) = 0$. □

It is easy to come up with examples of wandering sets in infinite measure spaces. For instance, taking $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $\varphi(x) := x + 1$, then any $W \subseteq [0, 1)$ is wandering. Also note that if there exists a wandering set W of non-zero measure, then there is a positive-measure set of points in W that are not recurrent with respect to W. As it turns out, the existence of non-trivial wandering sets is the only obstruction to recurrence. Interestingly, this works even without the measure-preserving property:

Theorem 26.3 (Halmos, 1947) *Let $(\mathscr{X}, \mathcal{G}, \mu)$ be a measure space and $\varphi\colon \mathscr{X} \to \mathscr{X}$ a measurable map. Then the following are equivalent:*

(1) *φ is* recurrent *in the sense*

$$\forall A \in \mathcal{G}\colon\; \mu\Bigl(A \smallsetminus \bigcup_{k\geqslant 1} \varphi^{-k}(A)\Bigr) = 0 \tag{26.4}$$

(2) φ *is* conservative *in the sense*

$$\forall A \in \mathcal{G}\colon\ \ A \text{ is wandering } \Rightarrow \mu(A) = 0 \tag{26.5}$$

(3) φ *is* incompressible *in the sense*

$$\forall A \in \mathcal{G}\colon\ \ \varphi^{-1}(A) \subseteq A \ \Rightarrow\ \mu\big(A \smallsetminus \varphi^{-1}(A)\big) = 0 \tag{26.6}$$

Moreover, if φ preimages null sets to null sets, i.e.,

$$\forall A \in \mathcal{G}\colon\ \mu(A) = 0 \ \Rightarrow\ \mu(\varphi^{-1}(A)) = 0 \tag{26.7}$$

then (1-3) are also equivalent to

(4) φ *is* infinitely recurrent *in the sense*

$$\forall A \in \mathcal{G}\colon\ \mu\bigg(A \smallsetminus \bigcap_{n\geqslant 1}\bigcup_{k\geqslant n}\varphi^{-k}(A)\bigg) = 0 \tag{26.8}$$

Proof. Observe that, given $A \in \mathcal{G}$, the set

$$W := A \smallsetminus \bigcup_{k\geqslant 1}\varphi^{-k}(A) \tag{26.9}$$

is wandering and, conversely, if A is a wandering set, then $W = A$. Hence, the existence of a non-trivial wandering set is equivalent to the measure in (26.4) not being zero for some A, proving the equivalence (1) $\Leftrightarrow$ (2).

Next, given any A, let $B := \bigcup_{k\geqslant 0}\varphi^{-k}(A)$ and note that

$$\varphi^{-1}(B) \subseteq B \quad \wedge \quad B \smallsetminus \varphi^{-1}(B) = A \smallsetminus \bigcup_{k\geqslant 1}\varphi^{-k}(A) \tag{26.10}$$

It follows that (3) $\Rightarrow$ (1). For the converse note that

$$\varphi^{-1}(A) \subseteq A \ \Rightarrow\ \bigcup_{k\geqslant 1}\varphi^{-1}(A) = \varphi^{-1}(A) \tag{26.11}$$

and so

$$\varphi^{-1}(A) \subseteq A \ \Rightarrow\ A \smallsetminus \varphi^{-1}(A) = A \smallsetminus \bigcup_{k\geqslant 1}\varphi^{-k}(A) \tag{26.12}$$

which now readily gives (1) $\Rightarrow$ (3).

For the equivalence of (4) with the rest, suppose (26.7). Note that the set in (26.4) are those $x \in A$ on which the iterations of φ never return to A while the set in (26.8) is the set of $x \in A$ on which the iterations of φ return to A only finitely often. Hence, trivially, (4) $\Rightarrow$ (1). For the converse we note

$$\begin{aligned}&\bigg(A \smallsetminus \bigcap_{n\geqslant 1}\bigcup_{k\geqslant n}\varphi^{-k}(A)\bigg) \smallsetminus \bigg(A \smallsetminus \bigcup_{k\geqslant 1}\varphi^{-k}(A)\bigg)\\ &\qquad \subseteq \bigcup_{n\geqslant 0}\bigg(\varphi^{-n}(A) \smallsetminus \bigcup_{k\geqslant n+1}\varphi^{-k}(A)\bigg) = \bigcup_{n\geqslant 0}\varphi^{-n}\bigg(A \smallsetminus \bigcup_{k\geqslant 1}\varphi^{-k}(A)\bigg)\end{aligned} \tag{26.13}$$

Under (1) and (26.7), each set in the union on the right is of zero measure, proving (1) $\Rightarrow$ (4) as desired. $\square$

Hereby we get:

Corollary 26.4 (Halmos' Recurrence Theorem) *Let $(\mathscr{X}, \mathcal{G}, \mu)$ be a measure space and let $\varphi\colon \mathscr{X} \to \mathscr{X}$ be a measurable map with φ^{-1} preserving null sets. Then φ is conservative if and only if*

$$\forall A \in \mathcal{G}\colon \quad \sum_{k \geqslant 0} 1_A \circ \varphi^k = \infty \ \mu\text{-a.e. on } A \tag{26.14}$$

Proof. This is (1) ⇔ (4) in Theorem 26.3 under (26.7). □

26.2 Hopf decomposition.

As our next observation we note that, assuming that φ is invertible with φ and φ^{-1} measurable and null-set preserving, the space $\mathscr{X}$ can always be partitioned into a part where φ acts conservatively and a part where it dissipative, (i.e., not recurrent on any positive-measure subset thereof):

Theorem 26.5 (Hopf decomposition) *Let $(\mathscr{X}, \mathcal{G}, \mu)$ be σ-finite and let $\varphi\colon \mathscr{X} \to \mathscr{X}$ be a bimesurable bijection such that φ and φ^{-1} preserve null sets. Then there exists $W \in \mathcal{G}$ such that*

$$W \text{ is wandering} \tag{26.15}$$

and, denoting,

$$D := \bigcup_{n \in \mathbb{Z}} \varphi^n(W) \quad \text{and} \quad C := \mathscr{X} \smallsetminus D \tag{26.16}$$

we have $\varphi(C) = C$ and $\varphi(D) = D$. Moreover,

$$\forall A \in \mathcal{G}\colon \ A \subseteq C \ \wedge \ A \text{ wandering} \ \Rightarrow \ \mu(A) = 0 \tag{26.17}$$

and so the restriction of φ to C is conservative.

The proof uses quite crucially that φ is invertible as a tool to get around the fact that the union of two wandering sets is not necessarily wandering. Indeed, we instead use the underlying structure to show that the union of the whole "doubly-infinite trajectory" of two wandering sets is the (doubly infinite) trajectory of a wandering set. This is the content of:

Lemma 26.6 *Let $(\mathscr{X}, \mathcal{G})$ be measure space and let $\varphi\colon \mathscr{X} \to \mathscr{X}$ be a bimesurable bijection. Given $W, W' \in \mathcal{G}$ wandering, set*

$$\widetilde{W} := W' \smallsetminus \bigcup_{k \in \mathbb{Z}} \varphi^k(W) \tag{26.18}$$

Then

$$W \cup \widetilde{W} \text{ is wandering} \tag{26.19}$$

and

$$\bigcup_{k \in \mathbb{Z}} \varphi^k(W \cup \widetilde{W}) = \Bigl(\bigcup_{k \in \mathbb{Z}} \varphi^k(W)\Bigr) \cup \Bigl(\bigcup_{k \in \mathbb{Z}} \varphi^k(W')\Bigr) \tag{26.20}$$

Proof. For $W, W' \in \mathcal{G}$ wandering, define $\widetilde{W}$ as above. Then $\widetilde{W}$, being a subset of a wandering set, is wandering. We also have

$$\forall k \neq 0\colon\;\; \widetilde{W} \cap \varphi^k(W) = \emptyset \;\wedge\; W \cap \varphi^k(\widetilde{W}) = \emptyset \tag{26.21}$$

where the first part is by definition of $\widetilde{W}$ and the second part follows from the first by applications of φ^{-1}. The distributive law for intersections and unions yields

$$\begin{aligned}(W \cup \widetilde{W}) &\cap \varphi^k(W \cup \widetilde{W})\\ &= \bigl[W \cap \varphi^k(W)\bigr] \cup \bigl[\widetilde{W} \cap \varphi^k(\widetilde{W})\bigr] \cup \bigl[W \cap \varphi^k(\widetilde{W})\bigr] \cup \bigl[\widetilde{W} \cap \varphi^k(W)\bigr]\end{aligned} \tag{26.22}$$

For $k \neq 0$, the first two sets on the right are empty because W and $\widetilde{W}$ are wandering and the other two sets are empty by (26.21). It follows that $W \cap \widetilde{W}$ is wandering.

For (26.20) we use the definition of $\widetilde{W}$ to note that

$$\forall k \in \mathbb{Z}\colon\;\; \varphi^k(\widetilde{W}) = \varphi^k(W') \smallsetminus \bigcup_{k\in\mathbb{Z}} \varphi^k(W) \tag{26.23}$$

This implies

$$\bigcup_{k\in\mathbb{Z}} \varphi^k(W \cup \widetilde{W}) = \Bigl(\bigcup_{k\in\mathbb{Z}} \varphi^k(W)\Bigr) \cup \Bigl(\bigcup_{k\in\mathbb{Z}} \varphi^k(W') \smallsetminus \bigcup_{k\in\mathbb{Z}} \varphi^k(W)\Bigr) \tag{26.24}$$

which gives (26.20). □

Proof of Theorem 26.5. As no non-null wandering sets exist when $\mu(\mathscr{X}) < \infty$, we may assume that $\mu(\mathscr{X}) = \infty$. Thanks to the σ-finiteness assumption, there are disjoint measurable sets $\{\mathscr{X}_n\}_{n\geqslant 0}$ such that $\mathscr{X} = \bigcup_{n\geqslant 0} \mathscr{X}_n$ and $0 < \mu(\mathscr{X}_n) < \infty$ for each $n \geqslant 0$. For $W \in \mathcal{G}$ set

$$\Phi(W) := \sum_{n\geqslant 0} 2^{-n} \frac{1}{\mu(\mathscr{X}_n)} \mu\Bigl(\mathscr{X}_n \cap \bigcup_{k\in\mathbb{Z}} \varphi^k(W)\Bigr) \tag{26.25}$$

and, noting that $\Phi(W) \leqslant 2$, let $\{W_k\}_{k\geqslant 0}$ be a sequence of wandering sets such that $\Phi(W_k)$ converges to

$$c := \sup\bigl\{\Phi(W)\colon W \in \mathcal{G} \;\wedge\; \text{wandering}\bigr\} \tag{26.26}$$

By Lemma 26.6 we may assume that $\{W_k\}_{k\geqslant 0}$ is non-decreasing so let

$$W := \bigcup_{k\geqslant 0} W_k \tag{26.27}$$

The Monotone Convergence Theorem then gives $\Phi(W) = c$. Next we claim that W is wandering. This follows by noting that, for $k \neq 0$,

$$W \cap \varphi^k(W) = \bigcup_{j,\ell\geqslant 0} W_j \cap \varphi^k(W_\ell) \subseteq \bigcup_{j,\ell\geqslant 0} W_{j\vee\ell} \cap \varphi^k(W_{j\vee\ell}) = \emptyset \tag{26.28}$$

where $j \vee \ell := \max\{j, \ell\}$ and where in the last step we use that each W_j is wandering. We conclude that W is a maximizer of (26.26).

Define D and C as above and let $W' \subseteq C$ be wandering. Then $\widetilde{W}$ defined in (26.18) obeys $\widetilde{W} = W'$. Using Lemma 26.6 it follows that $W \cup W'$ is wandering and that $\bigcup_{k\in\mathbb{Z}} \varphi^k(W')$ and $\bigcup_{k\in\mathbb{Z}} \varphi^k(W)$ are disjoint, and so

$$\Phi(W \cup W') = \Phi(W) + \Phi(W') \tag{26.29}$$

Since W is a maximizer, we must have $\Phi(W') = 0$ implying that $\mu(W') = 0$. Thus C contains no no-null wandering sets and $\varphi|_C$ is conservative. □

Theorem 26.5 gives us the partition

$$\mathscr{X} = C \cup D \quad \text{with} \quad C \cap D = \varnothing \tag{26.30}$$

where C is the *conservative* part of $\mathscr{X}$ and D is the *dissipative* (or *transient*) part of $\mathscr{X}$. For bimeasurably bijective measure-preserving transformations the dissipative part occurs only when $\mu(\mathscr{X}) = \infty$.

26.3 Stephanov-Hopf Ratio Ergodic Theorem.

Equipped with techniques to deal with infinite measure spaces, we now state another ergodic theorem whose main power rests exactly in this context:

Theorem 26.7 (Ratio Ergodic Theorem, Stephanov 1936, Hopf 1937) *Let $(\mathscr{X}, \mathcal{G}, \mu, \varphi)$ be a m.p.s and let $f, g \in L^1$ with $g \geqslant 0$. There exists a measurable map $R\colon \mathscr{X} \to \mathbb{R}$ with $R = R \circ \varphi$ such that*

$$\frac{\sum_{k=0}^{n-1} f \circ \varphi^k}{\sum_{k=0}^{n-1} g \circ \varphi^k} \underset{n\to\infty}{\longrightarrow} R \quad \text{a.e. on } \left\{ \sum_{k=0}^{\infty} g \circ \varphi^k = \infty \right\} \tag{26.31}$$

Moreover, $Rg \in L^1$ and if $\mu(\sum_{k=0}^{\infty} g \circ \varphi^k < \infty) = 0$ then $\int f \mathrm{d}\mu = \int Rg \mathrm{d}\mu$.

Before we delve into the proof, note that for μ finite, the choice $g := 1$ reduces to Birkhoff's Pointwise Ergodic Theorem. In this case we get that $R = \bar{f}/\bar{g}$, for $\bar{f}$ and $\bar{g}$ as in Theorem 22.1. The power of Theorem 26.7 thus really rests with μ infinite. To demonstrate this note that, for μ ergodic, Lemma 24.13 shows that $\bar{f} = 0$ and so Birkhoff's setting tells us nothing more than that $\sum_{k=0}^{n-1} f \circ \varphi^k$ is sublinear in n. The Ratio Ergodic Theorem tells us that this sum grows at the same rate for both f and g.

A number of different proofs of Theorem 26.7 exist in the literature. We will follow that of T. Kamae and M. Keane published in Osaka Journal of Mathematics in 1997. To explain the main idea, consider strictly positive sequences $\{a_n\}_{n\geqslant 0}$ and $\{b_n\}_{n\geqslant 0}$ and let

$$r := \liminf_{N\to\infty} \frac{b_0 + \cdots + b_{N-1}}{a_0 + \cdots + a_{N-1}} \tag{26.32}$$

Assuming $\sum_{k\geqslant 0} b_k = \infty$, the same r is obtained if the sums in the numerator and denominator start from the n-th term. It follows that for each $n \geqslant 0$ there is $m = m(n) \geqslant 0$ such that

$$\sum_{k=n}^{n+m-1}(r+\epsilon)a_k \geqslant \sum_{k=n}^{n+m-1} b_k \tag{26.33}$$

The problem with this conclusion is our lack of control of how large m is. This can be circumvented by replacing b_n by 0 when $m > M$ and keeping b_n as is otherwise. Denoting the sequence thus modified as $\{b'_n\}_{n\geqslant 0}$ we then have

$$\forall n \geqslant 0\, \exists m \in \{1,\dots,M\}\colon \quad \sum_{k=n}^{n+m-1}(r+\epsilon)a_k \geqslant \sum_{k=n}^{n+m-1} b'_k \tag{26.34}$$

This is what now directly feeds into the following lemma:

Lemma 26.8 *Let $\{a_n\}_{n\geqslant 0}$ and $\{b_n\}_{n\geqslant 0}$ be two non-negative sequences for which there exists $M \geqslant 1$ with the property*

$$\forall n \geqslant 0\ \exists m \in \{1,\dots,M\}\colon \quad \sum_{i=n}^{n+m-1} a_i \geqslant \sum_{i=n}^{n+m-1} b_i \tag{26.35}$$

Then

$$\forall N > M\colon \quad \sum_{i=0}^{N-1} a_i \geqslant \sum_{i=0}^{N-1} b_i \tag{26.36}$$

Proof. Define $\{n_k\}_{k\geqslant 0}$ recursively by $n_0 := 0$ and, for $k \geqslant 0$,

$$n_{k+1} := \inf\Biggl\{ m \geqslant n_k + 1\colon \sum_{i=n_k}^{m-1} a_i \geqslant \sum_{i=n_k}^{m-1} b_i \Biggr\} \tag{26.37}$$

The assumptions ensure

$$\forall k \geqslant 0\colon \quad n_{k+1} - n_k \in \{1,\dots,M\} \ \wedge \ \sum_{i=n_k}^{n_{k+1}-1} a_i \geqslant \sum_{i=n_k}^{n_{k+1}-1} b_i \tag{26.38}$$

Given $N > M$, set $k_N := \max\{k \geqslant 0\colon n_k < N\}$ and note that $N - M \leqslant n_{k_0} < N$. We now use positivity of a_i and b_i to get

$$\sum_{i=0}^{N-1} a_i \geqslant \sum_{k=0}^{k_N-1}\sum_{i=n_k}^{n_{k+1}-1} a_i \geqslant \sum_{k=0}^{k_N-1}\sum_{i=n_k}^{n_{k+1}-1} b_i \geqslant \sum_{i=0}^{N-M} b_i \tag{26.39}$$

thus giving us the desired claim. □

As a consequence of Lemma 26.8, from (26.34) we thus have

$$\forall N > M\colon \quad \sum_{k=0}^{N-1}(r+\epsilon)a_k \geqslant \sum_{k=0}^{N-M-1} b'_k \tag{26.40}$$

If we could pretend that b'_k equals b_k on the right, this would now bound the *limes superior* of the ratios in (26.32) by $r + \epsilon$, which in light of ϵ being arbitrary forces the existence of the limit (in extended $\mathbb{R}$). Of course, b'_k is different from b_k and so this does not quite work, but the actual proof is not far from that:

Proof of Theorem 26.7. For $h\colon \mathscr{X} \to \mathbb{R}$ abbreviate $h_n := \sum_{k=0}^{n-1} h\circ\varphi^k$. By decomposing f into positive and negative parts, it suffices to prove the claim for $f \geqslant 0$. Set

$$\mathscr{X}' := \Big\{ \sum_{k=0}^{\infty} g\circ\varphi^k = \infty \Big\} \tag{26.41}$$

and let

$$\underline{R}(x) := 1_{\mathscr{X}'}(x) \liminf_{n\to\infty} \frac{f_n(x)}{g_n(x)} \tag{26.42}$$

where the indicator takes care of the *limes inferior* being potentially undefined for $x \notin \mathscr{X}'$. Given any $\epsilon > 0$, we then have

$$\forall x\in\mathscr{X}'\colon\quad \underline{n}(x) := \inf\Big\{ n\geqslant 1\colon f_n(x) \leqslant \big[\underline{R}(x)+\epsilon\big] g_n(x)\Big\} < \infty \tag{26.43}$$

For $M \geqslant 1$, set

$$h(x) := \begin{cases} f(x), & \text{if } \underline{n}(x) \leqslant M \\ 0, & \text{else} \end{cases} \tag{26.44}$$

For each $x \in \mathscr{X}'$ we then have

$$\big[\underline{R}(x)+\epsilon\big] g_m(x) \geqslant h_m(x) \tag{26.45}$$

with $m := \underline{n}(x)$ if $\underline{n}(x) \leqslant M$ and $m := 1$ if $\underline{n}(x) > M$. Replacing x by $\varphi^n(x)$ while using that $\varphi^{-n}(\mathscr{X}') = \mathscr{X}'$ and that $\underline{R}\circ\varphi^n = \underline{R}$, it follows that, for all $x \in \mathscr{X}'$,

$$\forall n\geqslant 0\,\exists m = 1,\dots,M\colon\quad \sum_{i=n}^{n+m-1} \big[\underline{R}(x)+\epsilon\big] g\circ\varphi^i(x) \geqslant \sum_{i=n}^{n+m-1} h\circ\varphi^i(x) \tag{26.46}$$

Hence we get

$$\forall x\in\mathscr{X}'\,\forall N > M\colon\quad \big[\underline{R}(x)+\epsilon\big] g_N(x) \geqslant h_{N-M}(x) \tag{26.47}$$

by invoking Lemma 26.8.

Integrating (26.47) on $\mathscr{X}'$ we now obtain

$$\begin{aligned} N\int_{\mathscr{X}'} [\underline{R}+\epsilon] g\,\mathrm{d}\mu &= \int_{\mathscr{X}'} [\underline{R}+\epsilon] g_N\,\mathrm{d}\mu \\ &\geqslant \int_{\mathscr{X}'} h_{N-M}\,\mathrm{d}\mu = (N-M)\int_{\mathscr{X}'} h\,\mathrm{d}\mu \end{aligned} \tag{26.48}$$

where we used φ-invariance of $\mathscr{X}'$ and $\underline{R}$. For the last integral we note

$$\int_{\mathscr{X}'} h\,\mathrm{d}\mu \geqslant \int_{\mathscr{X}'} f\,\mathrm{d}\mu - \int f 1_{\{\underline{n}>M\}}\mathrm{d}\mu \tag{26.49}$$

Dividing (26.48) by N and taking $N\to\infty$, we combine these to

$$\int_{\mathscr{X}'} [\underline{R}+\epsilon] g\,\mathrm{d}\mu \geqslant \int_{\mathscr{X}'} f\,\mathrm{d}\mu - \int f 1_{\{\underline{n}>M\}} \tag{26.50}$$

We then conclude

$$\int_{\mathscr{X}'} \underline{R} g \,\mathrm{d}\mu \geqslant \int_{\mathscr{X}'} f \,\mathrm{d}\mu \tag{26.51}$$

by noting that $f, g \in L^1$ allows us to take $\epsilon \downarrow 0$ and $\underline{n} < \infty$ allows us to take $M \to \infty$ using Dominated Convergence Theorem.

We will now use pretty much the same reasoning to derive a converse inequality with $\overline{R}$ replaced by

$$\overline{R}(x) := 1_{\mathscr{X}'}(x) \limsup_{n\to\infty} \frac{f_n(x)}{g_n(x)} \tag{26.52}$$

For any $\epsilon \in (0,1)$ abbreviate $\overline{R}_\epsilon(x) := \max\{\epsilon^{-1}, \overline{R}(x)\}$ and note that then

$$\forall x \in \mathscr{X}'\colon \quad \overline{n}(x) := \inf\Big\{n \geqslant 1\colon f_n(x) \geqslant (1-\epsilon)\overline{R}_\epsilon(x) g_n(x)\Big\} < \infty \tag{26.53}$$

For $M \geqslant 1$, set

$$h(x) := \begin{cases} f(x), & \text{if } \underline{n}(x) \leqslant M \\ \epsilon^{-1} g(x), & \text{else} \end{cases} \tag{26.54}$$

Using that $(1-\epsilon)\overline{R}_\epsilon(x) \leqslant \epsilon^{-1}$ along with $\varphi^{-k}(\mathscr{X}') = \mathscr{X}'$ and $\overline{R}_\epsilon \circ \varphi = \overline{R}_\epsilon$ on $\mathscr{X}'$, for each $x \in \mathscr{X}'$ we now have

$$\forall n \geqslant 0\, \exists m = 1, \dots, M\colon \quad \sum_{i=n}^{n+m-1} h \circ \varphi^i(x) \geqslant \sum_{i=n}^{n+m-1} (1-\epsilon)\overline{R}_\epsilon(x) g \circ \varphi^i(x) \tag{26.55}$$

Lemma 26.8 then gives

$$\forall x \in \mathscr{X}'\, \forall N > 0\colon \quad h_{N+M}(x) \geqslant (1-\epsilon)\overline{R}_\epsilon(x)\, g_N(x) \tag{26.56}$$

Integrating we again turn this into

$$\begin{aligned} (N+M)\int_{\mathscr{X}'} h \,\mathrm{d}\mu &= \int_{\mathscr{X}'} h_{N+M} \,\mathrm{d}\mu \\ &\geqslant (1-\epsilon)\int_{\mathscr{X}'} \overline{R}_\epsilon\, g_N \,\mathrm{d}\mu = N(1-\epsilon)\int_{\mathscr{X}'} \overline{R}_\epsilon g \,\mathrm{d}\mu \end{aligned} \tag{26.57}$$

and for the integral on the left we get

$$\int_{\mathscr{X}'} h \,\mathrm{d}\mu \leqslant \int_{\mathscr{X}'} f \,\mathrm{d}\mu + \epsilon^{-1} \int g 1_{\{\overline{n} > M\}} \,\mathrm{d}\mu \tag{26.58}$$

Dividing (26.57) by N, taking $N \to \infty$ and then taking $M \to \infty$ followed by $\epsilon \downarrow 0$ with the help of Dominated and Monotone Convergence Theorems yields

$$\int_{\mathscr{X}'} f \mathrm{d}\mu \geqslant \int_{\mathscr{X}'} \overline{R} g \,\mathrm{d}\mu \tag{26.59}$$

which is complementary (with $\underline{R}$ replaced by $\overline{R}$) to (26.51).

Next observe that the φ-invariance of $\overline{R}$ implies $\int \overline{R} g \circ \varphi^k \mathrm{d}\mu = \int \overline{R} g \mathrm{d}\mu$. The bound (26.59) then gives

$$\forall n \geqslant 1\colon \quad \int_{\mathscr{X}'} \Big(\sum_{k=0}^{n} g \circ \varphi^k\Big) \overline{R} \,\mathrm{d}\mu < \infty \tag{26.60}$$

Since the sum is eventually positive at each point of $\mathscr{X}'$, we conclude that $\overline{R} < \infty$ a.e. on $\mathscr{X}'$. Next observe that the inequalities (26.51) and (26.59) imply

$$\int_{\mathscr{X}'} [\overline{R} - \underline{R}]\, g \,\mathrm{d}\mu = 0 \tag{26.61}$$

which (similarly as for (26.60)) with the help of non-negativity and φ-invariance of $\overline{R} - \underline{R}$ and the Monotone Convergence Theorem yields

$$\int_{\mathscr{X}'} [\overline{R} - \underline{R}] \Bigl(\sum_{k \geqslant 0} g \circ \varphi^k \Bigr) \mathrm{d}\mu = 0 \tag{26.62}$$

It follows that $\overline{R} = \underline{R}$ a.e. on $\mathscr{X}'$ and, since $\overline{R} < \infty$ a.e. on $\mathscr{X}'$, the convergence (26.31) takes place with $R := 1_{\{\overline{R} < \infty\}} \overline{R}$. The inequalities (26.51) and (26.59) give

$$\int_{\mathscr{X}'} f \mathrm{d}\mu = \int Rg \,\mathrm{d}\mu \tag{26.63}$$

and so $Rg \in L^1$. If $\mu(\sum_{k \geqslant 0} g \circ \varphi^k < \infty) = 0$, then we can drop $\mathscr{X}'$ from the first integral and get $\int f \mathrm{d}\mu = \int Rg \mathrm{d}\mu$. □

We remark that while the condition that $\sum_{k=0}^{\infty} g \circ \varphi^k = \infty$ is used crucially in the proof, the trivial fact that

$$\frac{\sum_{k=0}^{n-1} f \circ \varphi^k}{\sum_{k=0}^{n-1} g \circ \varphi^k} \underset{n \to \infty}{\longrightarrow} \frac{\sum_{k \geqslant 0} f \circ \varphi^k}{\sum_{k \geqslant 0} g \circ \varphi^k} \quad \text{on } \Bigl\{ 0 < \sum_{k=0}^{\infty} g \circ \varphi^k < \infty \Bigr\} \tag{26.64}$$

yields the convergence part of the claim as soon as $\sum_{k=0}^{\infty} g \circ \varphi^k > 0$ and $f \geqslant 0$. Note also that this special case arises only if $\mathscr{X}$ has a non-trivial dissipative part (i.e., a subset on which φ is transient).