25. Recurrence

Our next topic finds its motivation in Boltzmann's work. Recall that his (and Maxwell's) Ergodic Hypothesis asserts that the time average of a function tends, in the limit of large time intervals, to the average of f against a measure. In 1930s that convergence was proved to take place reducing the problem to the characterization of ergodic measures. However, several decades earlier, physicists P. Ehrenfest and his wife T. Afanassjewa wondered and wrote extensively about the reasons why such a statement should be true. Their conclusion was that this is because the trajectory of a Hamiltonian systems visits every point in the configuration space.

Such a statement was surprising and was soon found to be mathematically impossible, but a slight modification sounded reasonable: the trajectory comes arbitrarily close to every point. This is the so called *quasi-ergodic hypothesis* which is naturally linked with the concept of recurrence that we will address in this chapter.

25.1 Poincaré, Khinchin and Kac recurrence theorems.

We start with an old result that addresses the above problem:

Theorem 25.1 (Poincaré 1899) *Let $(\mathscr{X},\mathcal{G},\mu,\varphi)$ be a m.p.s. with $\mu(\mathscr{X})<\infty$. Then*

$$\forall A\in\mathcal{G}:\quad \mu\Big(\big\{x\in A\colon (\forall k\geqslant 1\colon \varphi^k(x)\notin A)\big\}\Big)=0 \tag{25.1}$$

In words, almost every $x\in A$ will return to A after a finite number of iterations of φ.

Proof. Denote

$$B:=\big\{x\in A\colon (\forall k\geqslant 1\colon \varphi^k(x)\notin A)\big\}=A\smallsetminus\bigcup_{k\geqslant 1}\varphi^{-k}(A) \tag{25.2}$$

and observe that $\varphi^{-1}(B)\subseteq\varphi^{-1}(A)$ and thus $B\cap\varphi^{-1}(B)=0$. Similarly we show that $\forall n\geqslant 1\colon \varphi^{-n}(B)\cap B=\varnothing$. Applying φ^{-k} on this we get

$$\forall n>k\geqslant 0\colon\ \varphi^{-n}(B)\cap\varphi^{-k}(B)=\varnothing \tag{25.3}$$

It follows that $\{\varphi^{-n}(B)\colon n\geqslant 0\}$ is a disjoint collection with $\mu(\varphi^{-n}(B))=\mu(B)$. Since $\mu(\mathscr{X})\geqslant\sum_{n\geqslant 0}\mu(\varphi^{-n}(B))$ we must have $\mu(B)=0$ whenever $\mu(\mathscr{X})<\infty$. □

This naturally leads to:

Definition 25.2 (Recurrent point) *Let $(\mathscr{X},\mathcal{G},\mu,\varphi)$ be a m.p.s. and $A\in\mathcal{G}$. A point $x\in\mathscr{X}$ is* recurrent with respect to A *if there exists $k\geqslant 1$ such that $\varphi^k(x)\in A$.*

The Poincaré Recurrence Theorem thus states that almost every point in A is recurrent with respect to A. Another way to phrase this is by introducing the *first return time to A*,

$$n_A(x):=\inf\big\{n\geqslant 1\colon \varphi^n(x)\in A\big\} \tag{25.4}$$

The Poincaré Recurrence Theorem then says

$$n_A<\infty \text{ a.e. on } A \tag{25.5}$$

Note that, as our example of $\varphi(x) = x + 1$ on $\mathbb{R}$ shows, this fails in infinite measure spaces; we will return to what may happen there later.

A natural question is how long will it take for a point to return to A. We have two theorems to discuss in this regard, the first of which focuses on returns of a set rather than a point itself:

Theorem 25.3 (Khinchin 1934) *Let $(\mathscr{X}, \mathcal{G}, \mu, \varphi)$ be a m.p.s. with $\mu(\mathscr{X}) = 1$. Then for all $A \in \mathcal{G}$ and all $\epsilon > 0$, the set*

$$E_\epsilon = \{n \in \mathbb{N} \colon \mu(\varphi^{-n}(A) \cap A) \geqslant \mu(A)^2 - \epsilon\} \tag{25.6}$$

has "bounded gaps" in the sense that

$$\exists n \geqslant 1\, \forall k \in \mathbb{N} \colon E_\epsilon \cap \{k, k+1, \dots, k+n-1\} \neq \varnothing \tag{25.7}$$

Proof. Consider the L^2-isometry defined by $Tf = f \circ \varphi$. Let $A \in \mathcal{G}$ and observe that

$$\mu(\varphi^{-n}(A) \cap A) = \langle T^n 1_A, 1_A \rangle \tag{25.8}$$

The Mean Ergodic Theorem implies that for each $\epsilon > 0$ there exists $n \geqslant 0$ such that

$$\left\| \frac{1}{n} \sum_{i=0}^{n-1} T^i 1_A - P 1_A \right\|_2 < \frac{\epsilon}{1 + \|1_A\|} \tag{25.9}$$

Since $TP = P$, applying T^k inside the norm gives

$$\left\| \frac{1}{n} \sum_{i=k}^{n+k-1} T^i 1_A - P 1_A \right\|_2 < \frac{\epsilon}{1 + \|1_A\|} \tag{25.10}$$

Invoking the Cauchy-Schwarz inequality, this implies

$$\left| \left\langle \frac{1}{n} \sum_{i=k}^{n+k-1} T^i 1_A - P 1_A, 1_A \right\rangle \right| < \frac{\epsilon \|1_A\|}{1 + \|1_A\|} \leqslant \epsilon \tag{25.11}$$

In order to rewrite the inner product, we note

$$\langle Pf, f \rangle = \langle Pf, Pf \rangle \geqslant \frac{\langle Pf, 1 \rangle^2}{\langle 1, 1 \rangle} \underset{P1=1}{=} \frac{\langle f, 1 \rangle^2}{\langle 1, 1 \rangle} \tag{25.12}$$

implying that

$$\langle P 1_A, 1_A \rangle \geqslant \frac{\langle 1_A, 1 \rangle^2}{\langle 1, 1 \rangle} \underset{\mu(\mathscr{X})=1}{=} \mu(A)^2 \tag{25.13}$$

Using this and (25.11) we now get

$$\begin{aligned} \frac{1}{n} \sum_{i=k}^{n+k-1} \mu(\varphi^{-k}(A) \cap A) &= \frac{1}{n} \sum_{i=k}^{n+k-1} \langle T^k 1_A, 1_A \rangle \\ &\geqslant \langle P 1_A, 1_A \rangle - \epsilon = \mu(A)^2 - \epsilon \end{aligned} \tag{25.14}$$

which proves the claim. □

Theorem 25.3 seems very strong; indeed, it says that a "neighborhood" of a trajectory of iterates of a m.p.t. will keep intersecting itself nearly regularly. However, one should not exaggerate the interpretation because the “neighborhood” can look very different every time it “comes back” — after all, no statement is made about how its points get scrambled around. This should be no surprise as all that goes into the proof of Theorem 25.3 is the measure-preserving property of φ.

Moving to quantifying the time it takes for a point to return to a set, we get:

Theorem 25.4 (Kac 1947) *Let $(\mathscr{X}, \mathcal{G}, \mu, \varphi)$ be a m.p.s. with $\mu(\mathscr{X}) < \infty$. Assume that φ admits a measurable inverse (which is then also a m.p.t.). Then*

$$\forall A \in \mathcal{G}\colon \int_A n_A \, \mathrm{d}\mu \leqslant \mu(\mathscr{X}) \tag{25.15}$$

If μ is ergodic and $\mu(A) > 0$, then equality holds.

Proof. Consider the sets $A_n := A \cap \{n_A = n\}$ and note that $n_A(x) = n$ implies $\varphi^k(x) \notin A$ for $k = 1, \dots, n-1$. Hence $\varphi^k(A_n) \cap A = \varnothing$ whenever $k = 1, \dots, n-1$ and so

$$\forall n, m \geqslant 1 \, \forall k = 1, \dots, n-1\colon \varphi^k(A_n) \cap A_m = \varnothing \tag{25.16}$$

We thus get that $\{\varphi^k(A_n)\colon 0 \leqslant k < n\}$ is a disjoint family. These sets are nicely visualized in the *Kakutani skyscraper* depicted in Fig. 1.

Noting that $\mu(\varphi^k(A_n)) = \mu(A_n)$, from (25.5) we obtain

$$\begin{aligned}\int n_A \, \mathrm{d}\mu &= \sum_{n \geqslant 1} n\mu(A_n) \\ &= \sum_{n \geqslant 1} \sum_{k=0}^{n-1} \mu\big(\varphi^k(A_n)\big) = \mu\biggl(\bigcup_{n \geqslant 1} \bigcup_{k=0}^{n-1} \varphi^k(A_n)\biggr) \leqslant \mu(\mathscr{X})\end{aligned} \tag{25.17}$$

proving the first part of the claim.

For the second part observe that, since $\varphi^n(A_n) \subseteq A$ and $A \subseteq \bigcup_{n \geqslant 1} \bigcup_{k=0}^{n-1} \varphi^k(A_n)$ a.e. (meaning that inclusion holds up to a null set), we have

$$\varphi\biggl(\bigcup_{n \geqslant 1} \bigcup_{k=0}^{n-1} \varphi^k(A_n)\biggr) \subseteq \bigcup_{n \geqslant 1} \bigcup_{k=0}^{n-1} \varphi^k(A_n) \quad \text{a.e.} \tag{25.18}$$

But the two sets have the same measure and so equality holds up to null sets. Denoting the set as S_A, we have $S_A \in \mathscr{I}'$ and $\int_A n_A \mathrm{d}\mu = \mu(S_A)$. Assuming $\mu(A) > 0$, the integral is strictly positive and so $\mu(S_A) > 0$. If μ is also ergodic, we must have $\mu(S_A) = \mu(\mathscr{X})$, proving that equality holds. □

The theorem is particularly useful in the ergodic case in which case we can rewrite (assuming $\mu(A) > 0$) the result as

$$\frac{1}{\mu(A)} \int_A n_A \, \mathrm{d}\mu = \Bigl(\frac{\mu(A)}{\mu(\mathscr{X})}\Bigr)^{-1} \tag{25.19}$$

This tells us that, conditional on starting from A, the expected time for the iterates of φ to return to A is equal to the inverse of the (proportional) mass of μ in A.

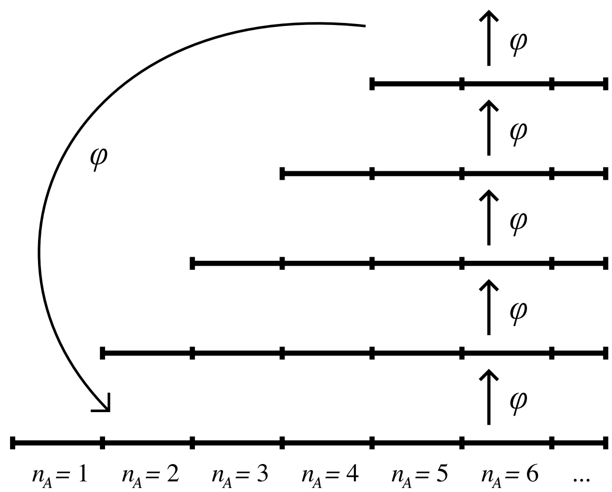

FIGURE 1. The Kakutani skyscaper for (non-null) set A with φ invertible. Here the "foundation" is the set A decomposed into "intervals" $A_n := A \cap \{n_A = n\}$. (We ignore the part where $n_A = \infty$ as it is null.) Iterations of φ on each A_n at the point keeps lifting it, one floor at the time, until a "patio" is hit, which φ maps back to A.

25.2 Some interesting consequences.

Let us present some interesting consequences of the above conclusions. The Poincaré Recurrence Theorem naturally leads to the concept of *induced transform* (a.k.a. *derivative transform*). This is the map $\varphi_A : A \to A$ defined for any $A \in \mathcal{G}$ with $\mu(A) > 0$ as

$$\varphi_A(x) := \begin{cases} \varphi^{n_A(x)}(x), & \text{if } n_A(x) < \infty \\ x, & \text{if } n_A(x) = \infty \end{cases} \tag{25.20}$$

where n_A is as in (25.4). Under $\mu(\mathscr{X}) < \infty$ and with φ an m.p.t., the Poincaré Recurrence Theorem ensures the validity of (25.5) and the second line in the definition above becomes a null-set proviso, which is desired as the map acts trivially in this case.

A key point about the induced transform is that it inherits the properties of φ:

Proposition 25.5 *Let μ be a finite measure on $(\mathscr{X}, \mathcal{G})$ and let $\varphi : \mathscr{X} \to \mathscr{X}$ be a map. Given $A \in \mathcal{G}$ such that $\mu(A) > 0$, define $\mathcal{G}_A := \{A \cap B : B \in \mathcal{G}\}$ and $\mu_A(B) := \mu(A \cap B)$. Then*

(1) *φ is $\mathcal{G}$-measurable $\Rightarrow$ φ_A is $\mathcal{G}_A$-measurable*
(2) *φ m.p.t. on $(\mathscr{X}, \mathcal{G}, \mu)$ $\Rightarrow$ φ_A m.p.t. on $(A, \mathcal{G}_A, \mu_A)$*
(3) *φ ergodic with respect to μ $\Rightarrow$ φ_A ergodic with respect to μ_A*

We leave the proof of this proposition to a homework assignment. The concept of the induced transform appears (implicitly or explicitly) throughout probability. To give an example, consider a stationary sequence $\{X_k\}_{k \geqslant 0}$ of $\{0,1\}$-valued random variables with $P(X_0 = 0) > 0$. Define $A := \{X_0 = 0\}$ and let θ be the left shift. Then θ_A is map representing the "shift to the nearest zero" which is well defined because, by the Poincaré Recurrence Theorem, $\{X_k\}_{k\geqslant 0}$ will contain infinitely many zero's a.s.

Our next observation concerns the arguments relying on the Kakutani skyscraper in the proof of Theorem 25.4. For convenience of expression, here we say that "$A \subseteq B$ a.e." to denote $\mu(A \smallsetminus B) = 0$ and "$A = B$ a.e." to denote $\mu(A \triangle B) = 0$.

Corollary 25.6 *Let $(\mathscr{X}, \mathcal{G}, \mu, \varphi)$ be a measurably-invertible m.p.s. with $\mu(\mathscr{X}) < \infty$ and, given $A \in \mathcal{G}$, let*

$$S_A := \bigcup_{n\geqslant 1} \bigcup_{k=0}^{n-1} \varphi^k\big(A \cap \{n_A = n\}\big) \tag{25.21}$$

be the set represented by the Kakutani skyscraper. Then $S_A \in \mathscr{I}'$ and $A \subseteq S_A$ a.e. Moreover, S_A is minimal with this property in the sense

$$\forall B \in \mathscr{I}'\colon\ \ A \subseteq B \text{ a.e.} \ \Rightarrow\ S_A \subseteq B \text{ a.e.} \tag{25.22}$$

In particular, μ ergodic implies $\mu(S_A) \in \{0, \mu(\mathscr{X})\}$.

Proof. All the properties have been already proved except (25.22). Suppose $A \subseteq B$ a.e. for $B \in \mathscr{I}'$. Then $A_n := A \cap \{n_A = n\} \subseteq B$ a.e. for each $n \geqslant 1$ and so $\varphi^k(A_n) \subseteq B$ a.e. for each $0 \leqslant k < n$. But then also $S_A \subseteq B$ a.e., as desired. □

The Kakutani skyscraper picture allows one to deduce that the whole space is, to within arbitrary small error, an orbit of a given length of a particular set:

Theorem 25.7 (Kakutani 1943, Rokhlin 1948) *Let $(\mathscr{X}, \mathcal{G}, \mu, \varphi)$ be an measurably-invertible ergodic m.p.s. with $\mu(\mathscr{X}) < \infty$. Assume that μ is non-atomic (i.e., $\forall A \in \mathcal{G}$ with $\mu(A) > 0$ there exists $B \in \mathcal{G}$ with $0 < \mu(B) < \mu(A)$). Then for each $\epsilon > 0$ and $m \geqslant 1$ there exists $B \in \mathcal{G}$ such that*

$$B, \varphi^{-1}(B), \dots, \varphi^{-m+1}(B) \text{ are disjoint} \tag{25.23}$$

and

$$\mu\bigg(\mathscr{X} \smallsetminus \bigcup_{k=0}^{m-1} \varphi^{-k}(B)\bigg) < \epsilon \tag{25.24}$$

The idea of the proof is to pick A of small measure and consider the Kakutani skyscraper associated with A. Then we take a suitable subset of its parts to be B so that the two properties hold true. We leave the details to a homework assignment.