MATH275C: HW#6

Due: Fri, June 13

Problem 1: For S a countable set, let $\{p_t(x,y)\colon t \geqslant 0, x,y \in S\}$ be a transition function and, given $x \in S$, let $c(x) := \lim_{t\downarrow 0}[1 - p_t(x,x)]/t$ be the instantaneous jump rate from x. Prove

$$\limsup_{t\downarrow 0} \sup_{y\neq x} p_t(y,x) < 1 \quad \Rightarrow \quad c(x) < \infty$$

(This gives a criterion for a state to be stable.) *Hint*: Follow an argument similar to the proof of existence of $q(x,y)$ but this time for $p_t(x,x)$ — which is what you need to lower-bound to get an upper bound on $c(x)$.

Problem 2: A "downward escalator" is a process X on $S := \{0,1,\dots\} \cup \{\infty\}$ defined as follows: Let $\{\alpha_k\}_{k\geqslant 1}$ be positive numbers with $\sum_{k\geqslant 1} \alpha_k^{-1} < \infty$. Set $\alpha_\infty := 1$ and let $T_1, \dots, T_\infty$ be independent with $T_k = \text{Exponential}(\alpha_k)$. For each $t \geqslant 0$ define

$$X_t := \inf\Big\{k \geqslant 0\colon T_\infty + \sum_{j\geqslant k+1} T_j \leqslant t\Big\}$$

Do as follows:

(1) Prove that $p_t(k,\ell) := P(X_{u+t} = \ell | X_u = k)$ is the same for all $u > 0$.
(2) Prove that p_t is a transition function on S with all states non-instanteneous.
(3) Prove that $q(k,\ell) := \frac{\mathrm{d}}{\mathrm{d}t} p_t(k,\ell)|_{t=0^+}$ is not a proper Q-matrix.

This supplies an example of a transition function without instantaneous states but with a defective state. The process X is Markov but since it has an accumulation point of jumps, it does not conform to our definition of continuous-time Markov chains.

Problem 3: LIGGETT, EX 2.31, PAGE 75

Problem 4: Suppose that $\{T_k\}_{k\geqslant 1}$ are i.i.d. exponentials and $\{Z_k\}_{k\geqslant 0}$ a path of an independent discrete-time Markov chain with transition kernel

$$\overline{\mathsf{P}}(x,y) := \begin{cases} \frac{q(x,y)}{c(x)} 1_{\{x\neq y\}}, & \text{if } c(x) > 0, \\ \delta_{x,y}, & \text{if } c(x) = 0, \end{cases}$$

for a Q-matrix $\{q(x,y)\colon x,y \in S\}$ and $c(x) := -q(x,x)$. Define

$$N(t) := \sup\Big\{m \geqslant 0\colon \sum_{k=1}^{m} T_k/c(Z_k) \leqslant t\Big\}$$

and suppose non-explosivity; i.e., $\forall x \in S \forall t \geqslant 0\colon P^x(N(t) < \infty) = 1$. Writing $X_t := Z_{N(t)}$ let $\mathcal{F}_t := \sigma(X_s, N(s)\colon s \leqslant t)$. Prove that

$$\forall t,s \geqslant 0\, \forall n \geqslant 0\colon P^x\big(N(t+s) - N(t) = n \,\big|\, \mathcal{F}_t\big) = P^{X_t}\big(N(s) = n\big), \quad \text{a.s.}$$

Then use this to prove that X is a continuous-time Markov chain per definition from class. Note: This is the construction on the way from Q-matrix to transition function during which we actually construct also the Markov chain.

Problem 5: LIGGETT, EX 2.38, PAGE 77
