MATH275C: HW#4

Due: Tue, May 27

Problem 1: Let $X_n := Z_1 + \dots + Z_n$ for $Z_1, Z_2, \dots$ i.i.d. $\mathbb{Z}$-valued with

$$P(Z_1 = n) = \frac{c}{n^2+1}, \quad n \in \mathbb{Z},$$

where c is a constant that makes this a probability mass function on $\mathbb{Z}$. Prove that X is recurrent. (This shows that the Chung-Fuchs criterion is not sharp.)

Problem 2: Consider the simple symmetric random walk X on $\mathbb{Z}$ and let P^x be the law for the walk started at x. We now "condition" on the walk to stay positive forever as follows: Set $\tau_y := \inf\{n \geqslant 0 \colon X_n = y\}$ and prove that

$$Q^x(A) := \lim_{n\to\infty} P^x(X \in A \,|\, \tau_n < \tau_0)$$

exists for all $x \geqslant 1$ and all events A depending only on finitely many coordinates. Then prove that Q^x is the law of a Markov chain that is an h-transform of the simple random walk by a suitable positive harmonic function h on $\mathbb{Z}_+$.

Problem 3: Let S be a rooted binary tree which we may identify with the set $\bigcup_{n\geqslant 0}\{0,1\}^n$ of finite binary sequences. The tree has a natural boundary $\partial S := \{0,1\}^{\mathbb{N}}$ given by the set of infinite binary sequences. Endow S with the metric

$$\rho(\sigma,\sigma') = \rho(\sigma',\sigma) := \sum_{i=1}^{m} 2^{-i}|\sigma_i - \sigma'_i| + \sum_{i=m+1}^{n} 2^{-i}$$

whenever $\sigma \in \{0,1\}^n$ and $\sigma' \in \{0,1\}^m$ with $m \leqslant n$ and extend it by suitable limits to ∂S. Note that $S \cup \partial S$ is compact and that ∂S may be identified with $[0,1)$. We write ϱ for the root, x for the vertices of S and α for points in ∂S below.

The tree is also a graph with any two sequences being neighbors whenever their length differs by one. Let X be the "random walk" on S which, technically, is a Markov chain with transition kernel $\mathsf{P}(x,\cdot)$ uniform over the neighbors of x. Do as follows:

(1) Prove that X is transient and that, for any $x \in S$,

$$X_\infty := \lim_{n\to\infty} X_n \text{ exists} \ \wedge\ X_\infty \in \partial S \quad P^x\text{-a.s.}$$

(2) Writing $\mathcal{L}_x$ for the law of X_∞ on ∂S under P^x defined by

$$\mathcal{L}_x(B) := P^x(X_\infty \in B)$$

for Borel $B \subseteq \partial S$, prove $\forall x \in S \colon \mathcal{L}_x \ll \mathcal{L}_\varrho \ \wedge\ \mathcal{L}_\varrho \ll \mathcal{L}_x$.

(3) Prove that for each bounded harmonic function h on S with $h(\varrho) = 1$ there exists a bounded Borel function $g_h \colon \partial S \to \mathbb{R}$ such that

$$\forall x \in S\colon \quad h(x) = \int_{\partial S} K(x,\alpha) g_h(\alpha) \mathcal{L}_\varrho(\mathrm{d}\alpha)$$

where $\mathsf{K}(x,\cdot) := \frac{\mathrm{d}\mathcal{L}_x}{\mathrm{d}\mathcal{L}_\varrho}$ is the Radon-Nikodym derivative of $\mathcal{L}_x$ w.r.t. $\mathcal{L}_\varrho$. Then prove that

$$h(X_n) \underset{n\to\infty}{\longrightarrow} g_h(X_\infty) \quad P^x\text{-a.s.}$$

(4) Construct a bounded harmonic function that alternates between 0 and 1 on a path to infinity, thus showing that (in spite of the above convergence along paths of the chain) such functions may not extend continuously to ∂S.

(5) Construct a positive harmonic function h such that, under the h-transformed measure P_h^x, we have

$$X_n \underset{n\to\infty}{\longrightarrow} \underline{0} \quad P_h^x\text{-a.s.}$$

where $\underline{0}$ is the sequence $(0, 0, \dots) \in \partial S$.

The statement in (3) is a variant of *Poisson-kernel* representation of bounded harmonic functions by their boundary values. For unbounded non-negative functions, this is achieved by way of the Martin boundary, which by (5) coincides with ∂S.