MATH275C: HW#3

Due Mon May 5

Problem 1: Prove Rényi's theorem: A m.p.t. φ on a probability space $(\mathscr{X}, \mathcal{G}, \mu)$ is strongly mixing if and only if

$$\forall A \in \mathcal{G}\colon \quad \lim_{n\to\infty} \mu\big(\varphi^{-n}(A) \cap A\big) = \mu(A)^2$$

Hint: Consider the orthogonal projection of 1_B onto $\operatorname{span}(\{1\} \cup \{1_A \circ \varphi^k \colon k \geqslant 0\})$

Problem 2: Let $\{X_k\}_{k\geqslant 0}$ be i.i.d. $\mathbb{Z}^d$-valued random variables. For each $n \geqslant 1$ denote $S_n := X_1 + \cdots + X_n$ and set

$$R_n := |\{S_0, \dots, S_{n-1}\}|$$

Prove that

$$\lim_{n\to\infty} \frac{R_n}{n} = P\big(\forall n \geqslant 1\colon S_n \neq 0\big) \quad \text{a.s.}$$

Prove that the r.h.s. vanishes in $d = 1$ when $X_1 \in L^p$ for some $p > 1$ and $EX_1 = 0$.

Problem 3: DURRETT: EX 5.1.1, PAGE 272

Problem 4: DURRETT: EX 5.1.4, PAGE 272

Problem 5: Consider an irreducible, recurrent Markov chain $\{X_n\}_{n\geqslant 0}$ on a countable state space S. This ensures positivity and finiteness of $\nu_x(y) := E^x\big(\sum_{0\leqslant k<T_x} 1_{\{X_k=y\}}\big)$ for all $x, y \in S$, where $T_x := \inf\{n \geqslant 1\colon X_n = x\}$. Prove that

$$\forall x, y, z \in S\colon \quad \nu_x(y)\nu_y(z) = \nu_x(z)$$

and

$$\forall x, y \in S\colon \quad \nu_x(y) = \frac{P^x(T_y < T_x)}{P^y(T_x < T_y)}$$

Problem 6: A chess piece moves around on the 8×8 chessboard by taking, at each time, one of its allowed moves uniformly at random. Determine whether the associated Markov chain is irreducible and find its stationary distributions. Do this for (1) a pawn, (2) a rook, (3) a knight. For the knight, determine the expected return time to the corner for the chain started in that corner.
