MATH275C: HW#2

Due Mon Apr 21

This homework practices the concepts of measure-preserving transformation (m.p.t.) and ergodicity. Problem 3 connects to the subject via the assumption made in Problem 4.

Problem 1: Let $(\mathscr{X}, \mathcal{G}, \mu, \varphi)$ be a m.p.s. and set $\mathscr{I} := \{A \in \mathcal{G} : \varphi^{-1}(A) = A\}$. Let (S, Σ) be a measurable space. Given a measurable $f : \mathscr{X} \to S$ such that $f \circ \varphi = f$ μ-a.e., prove that there exists g such that $f = g$ μ-a.e. and g is $\mathscr{I}$-measurable.

Hint: First think of an argument for $(S, \Sigma) := (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Problem 2: Let μ be a finite measure on $(\mathscr{X}, \mathcal{G})$ and let $\varphi : \mathscr{X} \to \mathscr{X}$ be a bimesurably invertible map. Given $A \in \mathcal{G}$ such that $\mu(A) > 0$, define $\mathcal{G}_A := \{A \cap B : B \in \mathcal{G}\}$ and $\mu_A(B) := \mu(A \cap B)$ and let $\varphi_A : A \to A$ be defined by

$$\varphi_A(x) := \begin{cases} \varphi^{n_A(x)}(x), & \text{if } n_A(x) < \infty \\ x, & \text{if } n_A(x) = \infty \end{cases}$$

where $n_A(x) := \inf\{n \geqslant 1 : \varphi^n(x) \in A\}$. Prove the following:

(1) φ is $\mathcal{G}$-measurable $\Rightarrow$ φ_A is $\mathcal{G}_A$-measurable
(2) φ m.p.t. on $(\mathscr{X}, \mathcal{G}, \mu)$ $\Rightarrow$ φ_A m.p.t. on $(A, \mathcal{G}_A, \mu_A)$
(3) φ ergodic with respect to μ $\Rightarrow$ φ_A ergodic with respect to μ_A

Problem 3: Let μ be a finite measure on $(\mathscr{X}, \mathcal{G})$. Prove that $\{\mu(A) : A \in \mathcal{G}\}$ is a closed subset of $[0, \mu(\mathscr{X})]$. Note that, for μ non-atomic, which means that each $A \in \mathcal{G}$,

$$\mu(A) > 0 \;\Rightarrow\; \exists B \in \mathcal{G} :\; B \subseteq A \,\wedge\, 0 < \mu(B) < \mu(A)$$

we have $\{\mu(A) : A \in \mathcal{G}\} = [0, \mu(\mathscr{X})]$. Conclude also that each non-atomic finite measure μ has the property

$$\forall A \in \mathcal{G}\, \forall \theta \in [0,1]\, \exists B \in \mathcal{G} :\; B \subseteq A \,\wedge\, \mu(B) = \theta \mu(A)$$

i.e., we can split A measurably into two parts with given proportions of total mass.

Problem 4: Let $(\mathscr{X}, \mathcal{G}, \mu, \varphi)$ be an ergodic m.p.s. with $\mu(\mathscr{X}) < \infty$. Assume that

$$\big\{\mu(A) : A \in \mathcal{G}\big\} = \big[0, \mu(\mathscr{X})\big]$$

Prove that for each $\epsilon > 0$ and $m \geqslant 1$ there exists $B \in \mathcal{G}$ such that

$$B, \varphi^{-1}(B), \dots, \varphi^{-m+1}(B) \text{ are disjoint}$$

and

$$\mu\left(\mathscr{X} \smallsetminus \bigcup_{k=0}^{m-1} \varphi^{-k}(B) \right) < \epsilon$$

Note: This is usually presented under the assumption that μ is non-atomic. However, Problem 3 is not needed in either version. It was assigned for its independent interest.
