MATH275C: HW#1

Due Mon Apr 14

This homework practices the concepts of measure-preserving transformation (m.p.t.), ergodicity and the techniques underlying proofs of convergence in ergodic theorems.

Problem 1: (Boole transform) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ where λ is the Lebesgue measure. Define $\varphi\colon \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) := \begin{cases} x - 1/x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Prove that φ is an m.p.t. Conclude that

$$\forall f \in L^1\colon \quad \int_{\mathbb{R}} f(x)\mathrm{d}x = \int_{\mathbb{R}} f(x - 1/x)\mathrm{d}x$$

(This is what was discovered by G. Boole in 1857.)

Problem 2: (Random ergodic theorem) Let $(\mathscr{X}, \mathcal{G}, \mu)$ be a measure space, (S, Σ) a measurable space and $\varphi\colon \mathscr{X} \times S \to \mathscr{X}$ a map such that

(1) for each $y \in S$, the map $x \mapsto \varphi(x, y)$ is a m.p.t. on $(\mathscr{X}, \mathcal{G}, \mu)$,
(2) $(x, y) \mapsto \varphi(x, y)$ is $\mathcal{G} \otimes \Sigma/\mathcal{G}$-measurable.

Abbreviate the n-fold composition with second coordinates $y_0, \dots, y_{n-1}$ as

$$\varphi_n(x, y_0, \dots, y_{n-1}) := \varphi(\cdot, y_{n-1}) \circ \cdots \circ \varphi(\cdot, y_0)(x)$$

Let $\{Y_k\}_{k \geqslant 0}$ be i.i.d. S-valued random variables on $(\Omega, \mathcal{F}, P)$. Prove that for each function $f \in L^1(\mathscr{X}, \mathcal{G}, \mu)$ there exists $\bar{f} \in L^1(\mathscr{X} \times \Omega, \mathcal{G} \otimes \mathcal{F}, \mu \otimes P)$ such that

$$\frac{1}{n} \sum_{k=1}^{n} f \circ \varphi_k(\cdot, Y_0, \dots, Y_{k-1}) \underset{n \to \infty}{\longrightarrow} \bar{f}, \quad \mu\text{-a.e.}$$

holds for P-a.e. sample of $\{Y_k\}_{k \geqslant 0}$.

Problem 3: Let φ be a m.p.t. on $(\mathscr{X}, \mathcal{G}, \mu)$ with $\mu(\mathscr{X}) < \infty$. Prove that for each $f \in L^1(\mu)$ there exists a μ-null set $\mathcal{N} \in \mathcal{G}$ such that

$$\forall x \notin \mathcal{N}\ \forall \theta \in \mathbb{R}\colon \quad \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathrm{e}^{2\pi \mathrm{i} \theta k} f \circ \varphi^k(x) \text{ exists}$$

(The main point is that the convergence holds for all θ simultaneously.)

Problem 4: Let φ be a m.p.t. on $(\mathscr{X}, \mathcal{G}, \mu)$ with $\mu(\mathscr{X}) < \infty$ and let $f\colon \mathscr{X} \to \mathbb{R}$ be measurable with $f \geqslant 0$. Prove that

$$\varphi \text{ ergodic} \ \wedge \ \int f \mathrm{d}\mu = \infty \quad \Rightarrow \quad \frac{1}{n} \sum_{k=0}^{n-1} f \circ \varphi^k \underset{n \to \infty}{\longrightarrow} \infty \text{ a.s.}$$

Then give an example of the above setting with φ non-ergodic for which the ergodic averages have a finite limit a.s. yet $\int f \mathrm{d}\mu = \infty$.
