

Finishing Lebesgue integral

Last time: $\int f d\mu$ for $f: \Omega \rightarrow [0, \infty]$ as lower integral.
proving additivity:

Lemma 1 $\forall f, g: \Omega \rightarrow [0, \infty]$ measurable s.t.

$$\exists M > 0: f, g \leq M \wedge \mu(\{f > 0\} \cup \{g > 0\}) < \infty$$

The $\int (f+g) d\mu = \int f d\mu + \int g d\mu$

Lemma 2 Let $f: \Omega \rightarrow [0, \infty]$ be meas. Then

1) $\int (f \wedge M) d\mu \xrightarrow{M \rightarrow \infty} \int f d\mu$

2) $\forall \{A_n\}_{n \in \mathbb{N}}: A_n \uparrow \wedge \bigcup_{n \in \mathbb{N}} A_n \supseteq \{f > 0\} \Rightarrow \int f 1_{A_n} d\mu \rightarrow \int f d\mu$

Lemma (Markov's inequality) Let $f: \Omega \rightarrow [0, \infty]$. Then

$$\forall t > 0: \mu(f > t) \leq \frac{1}{t} \int f d\mu.$$

Pf: $t 1_{\{f > t\}} \leq f \Rightarrow \int t 1_{\{f > t\}} d\mu \leq \int f d\mu. \quad \square$

Now we use these to prove additivity ...

Pf of additivity let $f, g: \Omega \rightarrow [0, \infty]$ be meas. Know: $\int (f+g) d\mu \geq \int f d\mu + \int g d\mu$.
 Suffices to deal with case when $\int f d\mu < \infty \wedge \int g d\mu < \infty$.
 Hence $\forall t > 0$: $\mu(f+g > 2t) \leq \mu(f > t) + \mu(g > t) \leq \frac{1}{t} \int f d\mu + \frac{1}{t} \int g d\mu < \infty$.

$$\text{So } \int (f+g) \wedge M \cdot \mathbb{1}_{\{f+g > 2/n\}} \leq \int (f \wedge M + g \wedge M) \mathbb{1}_{\{f+g > 2/n\}} \stackrel{\text{Lemma 1}}{=} \int (f \wedge M) \mathbb{1}_{\{f+g > 2/n\}} + \int (g \wedge M) \mathbb{1}_{\{f+g > 2/n\}}$$

$$\stackrel{\text{monotonicity}}{\leq} \int f d\mu + \int g d\mu$$

Now take $n \rightarrow \infty$ and $M \rightarrow \infty$ to get (by Lemma 2)

$$\int (f+g) d\mu \leq \int f d\mu + \int g d\mu. \quad \square$$

Convergence statements.

Thm. (Monotone Convergence Thm) Let $\{f_n\}_{n \in \mathbb{N}}$, f be measurable functions $\Omega \rightarrow [0, \infty]$ s.t.
 1) $\forall n \in \mathbb{N}$: $f_n \leq f_{n+1}$ 2) $f = \lim_{n \rightarrow \infty} f_n$. Then $\int f_n d\mu \rightarrow \int f d\mu$.

Pf: $f_n \leq f$ so $\int f_n d\mu \leq \int f d\mu$ and thus $\lim_{n \rightarrow \infty} \int f_n d\mu \leq \int f d\mu$. WLOG: $\sup_{n \in \mathbb{N}} \int f_n d\mu < \infty$.

Since $\{f > t\} = \bigcup_{n \in \mathbb{N}} \{f_n > t\}$ so $\mu(f > t) = \lim_{n \rightarrow \infty} \mu(f_n > t) \leq \frac{1}{t} \sup_{n \in \mathbb{N}} \int f_n d\mu < \infty$

$$\text{Next } (f \wedge M) \mathbb{1}_{\{f > t\}} \leq f_n + \varepsilon \mathbb{1}_{\{f > t\}} + M \mathbb{1}_{\{f > t\} \cap \{f - f_n > \varepsilon\}}.$$

$$\text{So } \int (f \wedge M) \mathbb{1}_{\{f > t\}} d\mu \leq \int f_n d\mu + \varepsilon \mu(f > t) + M \mu(\{f > t\} \cap \{f - f_n > \varepsilon\})$$

$\downarrow \emptyset$, finite measure so $\xrightarrow[n \rightarrow \infty]{\text{MCT}} 0$

$$\int (f \wedge M) \mathbb{1}_{\{f > t\}} d\mu \leq \lim_{n \rightarrow \infty} \int f_n d\mu$$

Now take $t \downarrow 0$, $M \rightarrow \infty$ to get

$$\int f d\mu \leq \lim_{n \rightarrow \infty} \int f_n d\mu \quad \square$$

Corollary Let $\{f_n\}_{n \in \mathbb{N}}$ be meas. functions $\Omega \rightarrow [0, \infty]$. Then

$$\sum_{n \in \mathbb{N}} \int f_n d\mu = \int \left(\sum_{n \in \mathbb{N}} f_n \right) d\mu$$

Corollary Let $f: \Omega \rightarrow [0, \infty]$ be meas. on $(\Omega, \mathcal{F})$.

Then $\nu(A) := \int f 1_A d\mu$ is a measure on $(\Omega, \mathcal{F})$.

Notation: $\int_A f d\mu := \int f 1_A d\mu$.

Pf: $g_n = \sum_{k=1}^n f_k \uparrow g = \sum_{n \in \mathbb{N}} f_n$
 $\sum_{k=1}^n \int f_k d\mu = \int g_n d\mu \xrightarrow{\text{MCT}} \int g d\mu = \int \sum_{n \in \mathbb{N}} f_n d\mu$ $\square$

Pf: $\{A_n\}_{n \in \mathbb{N}} \in \mathcal{F}^{\mathbb{N}}$ disjoint, set $f_n = 1_{A_n}$ above and note $\sum_{n \in \mathbb{N}} f_n = 1_{\bigcup_{n \in \mathbb{N}} A_n}$
 to get $\sum_{n \in \mathbb{N}} \nu(A_n) = \nu(\bigcup_{n \in \mathbb{N}} A_n)$ $\square$

Lemma (Fatou) Let $\{f_n\}_{n \in \mathbb{N}}$ be meas. $\Omega \rightarrow [0, \infty]$ functions.

Then $\int \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu$ Key! Need all $f_n \geq 0$!

Pf: $\forall m \geq 1 \forall k \geq m: \inf_{n \geq m} f_n \leq f_k$ so $\int \inf_{n \geq m} f_n d\mu \leq \inf_{k \geq m} \int f_k d\mu$
 Since $\inf_{n \geq m} f_n \uparrow \liminf_{n \rightarrow \infty} f_n$, we can apply MCT. $\square$

Signed integral

Def Let $f: \Omega \rightarrow \overline{\mathbb{R}}$ be measurable. If $\min \left\{ \int f^+ d\mu, \int f^- d\mu \right\} < \infty$, we say that f is integrable w.r.t. μ and set

$$\int f d\mu := \int f^+ d\mu - \int f^- d\mu$$

Lemma Let $f, g: \Omega \rightarrow \overline{\mathbb{R}}$ be measurable and such that f, g is integrable w.r.t. μ and

$$\int (f+g) d\mu = \int f d\mu + \int g d\mu.$$

$\int f d\mu, \int g d\mu$ exist and $\int f d\mu + \int g d\mu$
 $\Rightarrow$ NOT type $\infty - \infty$.

Pf Check cases. $\square$

Corollary: $\forall f, g: \Omega \rightarrow \overline{\mathbb{R}}$ measurable $\forall \lambda \in \mathbb{R}$:

$$1) \int \lambda f d\mu = \lambda \int f d\mu$$

$$2) f \leq g \Rightarrow \int f d\mu \leq \int g d\mu$$

Pf 1) $(\lambda f)^+ = \begin{cases} \lambda f^+ & \lambda > 0 \\ (-\lambda) f^- & \lambda < 0 \\ 0 & \lambda = 0 \end{cases}$ so $\int (\lambda f) d\mu = \int (\lambda f)^+ d\mu - \int (\lambda f)^- d\mu \underset{\lambda < 0}{=} -\lambda \int f^- d\mu + \lambda \int f^+ d\mu = \lambda \int f d\mu$

2) $g = f + \underbrace{g-f}_{\geq 0}$ so $\int g d\mu \geq \int f d\mu \quad \square$

Thm (Dominated Convergence Thm)

Let $\{f_n\}_{n \in \mathbb{N}}$, f be meas. $\Omega \rightarrow \mathbb{R}$ s.t. $\exists g: \Omega \rightarrow [0, \infty]$ with:

1) $f_n \rightarrow f$ pointwise

2) $\forall n \in \mathbb{N}: |f_n| \leq g$

3) $\int g \, d\mu < \infty$

Then $\int f_n \, d\mu \rightarrow \int f \, d\mu$

Pf Next time.

Note counterexamples? $\Omega = \mathbb{R}$, $\mu = \text{Lebesgue}$

1) $f_n := n \cdot 1_{(0, 1/n]}$

2) $f_n := \frac{1}{n} 1_{(-n, n)}$

These have no finitely integrable dominating function!