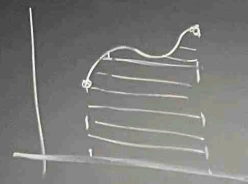


Lebesgue integral

Motivation:



Riemann



Lebesgue (needs measure)

Key idea: Approximate function by that with finitely many values.

For Riemann these are step functions

For Lebesgue — " — simple function

Want: $\varphi = \sum_{i=1}^n a_i 1_{A_i} \mapsto \int \varphi d\mu := \sum_{i=1}^n a_i \mu(A_i)$

Need:

Lemma: $\forall n, m \geq 1 \quad \forall a_1, \dots, a_n, b_1, \dots, b_m \geq 0 \quad \forall A_1, \dots, A_n, B_1, \dots, B_m \in \mathcal{F}$:

$$\sum_{i=1}^n a_i 1_{A_i} = \sum_{j=1}^m b_j 1_{B_j} \Rightarrow \sum_{i=1}^n a_i \mu(A_i) = \sum_{j=1}^m b_j \mu(B_j)$$

If wlog, $a_i, b_j \neq 0$.
 Then LHS = $\sum_{i=1}^n \sum_{j=1}^m a_i 1_{A_i \cap B_j}$ and $\mu(A_i) = \sum_{j=1}^m \mu(A_i \cap B_j)$, $\mu(B_j) = \sum_{i=1}^n \mu(A_i \cap B_j)$
 Note that then $A_i \cap B_j \neq \emptyset \Rightarrow a_i = b_j$
 (assuming that $\{A_i\}$ and $\{B_j\}$ disjoint)

Now: $\sum_{i=1}^n a_i \mu(A_i) \stackrel{\text{def}}{=} \sum_{i=1}^n \sum_{j=1}^m a_i \mu(A_i \cap B_j) = \sum_{i=1}^n \sum_{j=1}^m b_j \mu(A_i \cap B_j) = \sum_{j=1}^m b_j \mu(B_j)$. $\square$

Def: For $\varphi = \sum_{i=1}^n a_i 1_{A_i}$ simple, $\varphi \geq 0$: $\int \varphi d\mu := \sum_{i=1}^n a_i \mu(A_i)$.

Lemma $\forall \varphi, \psi$ simple, $\varphi, \psi \geq 0$:

- 1) $\varphi \leq \psi \Rightarrow \int \varphi d\mu \leq \int \psi d\mu$
- 2) $\int (\varphi + \psi) d\mu = \int \varphi d\mu + \int \psi d\mu$.

Pf: Find representation of φ, ψ using common $\{A_i\}_{i=1}^n$, check the formulas. $\square$

Extension to all ≥ 0 functions:

Def If $f: \Omega \rightarrow [0, \infty]$, set $\int f d\mu := \sup \left\{ \int \varphi d\mu : \varphi \text{ simple} \wedge 0 \leq \varphi \leq f \right\}$,
(analogue of lower Darboux integral)

Lemma $\forall f, g \in [0, \infty]^{\Omega} \forall \lambda \geq 0$:

- 1) $f \leq g \Rightarrow \int f d\mu \leq \int g d\mu$
- 2) $\int (\lambda f) d\mu = \lambda \int f d\mu$ ($0 \cdot \infty = 0$)
- 3) $\int (f+g) d\mu \geq \int f d\mu + \int g d\mu$

Pf: 1) $\varphi \leq f \Rightarrow \varphi \leq g$

2) WLOG $\lambda > 0$. Then $\varphi \leq f \Leftrightarrow \lambda \varphi \leq \lambda f$

3) Let $\{\varphi_n\}, \{\psi_n\}$ be simple st. $\varphi_n \leq f, \psi_n \leq g$, $\int \varphi_n d\mu \rightarrow \int f d\mu$, $\int \psi_n d\mu \rightarrow \int g d\mu$

Then $\int (f+g) d\mu \geq \int (\varphi_n + \psi_n) d\mu$
 $\stackrel{f+g \geq \varphi_n + \psi_n}{=} \int \varphi_n d\mu + \int \psi_n d\mu \rightarrow \int f d\mu + \int g d\mu$ $\square$

Ex $\mathcal{F} = \{\emptyset, \Omega\}$, let $A \subseteq \Omega$ be s.t. $A, A^c \neq \emptyset$
 Let $f := 1_A$, $g := 1_{A^c}$. Then $\int f d\mu = 1 = \int g d\mu$, yet $\int (f+g) d\mu = \int 1 d\mu = \mu(\Omega)$.

Punchline: will need to assume f to be measurable!

Thm (additivity) Suppose $f, g: \Omega \rightarrow [0, \infty]$ are $\mathcal{F}/\mathcal{B}(\mathbb{R})$ -measurable.

Then
$$\int (f+g) d\mu = \int f d\mu + \int g d\mu$$

Lemma Let $f, g: \Omega \rightarrow [0, \infty]$ be measurable and s.t.

$$\exists M > 0: f, g \leq M \wedge \mu(\{f > 0\} \cup \{g > 0\}) < \infty.$$

Then
$$\int (f+g) d\mu = \int f d\mu + \int g d\mu$$

Pf Let $n \geq 1$ be s.t. $2^n > M$. Set
$$\varphi_n := \sum_{k=0}^{2^n-1} k 2^{-n} \mathbf{1}_{\{k 2^{-n} \leq f < (k+1) 2^{-n}\}}$$

$$\tilde{\varphi}_n := \sum_{k=0}^{2^n-1} k 2^{-n} \mathbf{1}_{\{k 2^{-n} \leq g < (k+1) 2^{-n}\}}.$$

Note $\varphi, \tilde{\varphi}$ are simple s.t.

$$\varphi_n \leq f \leq \varphi_n + 2^{-n} \mathbf{1}_{\{f > 0\}}$$

$$\tilde{\varphi}_n \leq g \leq \tilde{\varphi}_n + 2^{-n} \mathbf{1}_{\{g > 0\}}$$

$$\begin{aligned} \text{Then } \int (f+g) d\mu &\leq \int (\varphi_n + \tilde{\varphi}_n + 2^{-n} 1_{f>0} + 2^{-n} 1_{g>0}) d\mu \\ &\stackrel{\text{add. for simple}}{=} \int \varphi_n d\mu + \int \tilde{\varphi}_n d\mu + 2^{-n} \mu(f>0) + 2^{-n} \mu(g>0) \\ &\leq \int f d\mu + \int g d\mu + 2^{-n+1} \mu(\{f>0\} \cup \{g>0\}) \xrightarrow{n \rightarrow \infty} \int f d\mu + \int g d\mu. \quad \square \end{aligned}$$

Lemma: Let $f, g: \Omega \rightarrow [0, \infty]$ be meas. Then

$$\begin{aligned} 1) \quad \int (f \wedge M) d\mu &\xrightarrow{M \rightarrow \infty} \int f d\mu \\ 2) \quad \forall \{A_n\}_{n \in \mathbb{N}} \in \mathcal{G}^{\mathbb{N}} \quad \forall A \in \mathcal{G}^{\mathbb{N}}: A_n \uparrow A \supseteq \{f>0\} &\Rightarrow \int f 1_{A_n} d\mu \xrightarrow{n \rightarrow \infty} \int f d\mu. \end{aligned}$$

$$(f \wedge M := \min\{f, M\})$$

Pf 1) Let $\varphi \geq 0$ be simple, $0 \leq \varphi \leq f$. Since $\max \varphi < \infty$, $\exists M_0 > 0 \forall M \geq M_0: \varphi \leq f \wedge M$.

So $\lim_{M \rightarrow \infty} \int (f \wedge M) d\mu \geq \int \varphi d\mu$. Take supremum over φ to get

$$\int f d\mu \geq \lim_{M \rightarrow \infty} \int (f \wedge M) d\mu \geq \int f d\mu.$$

2) $0 \leq \varphi \leq f$ simple, $\varphi = \sum_{i=1}^m a_i 1_{B_i}$. Then $\varphi 1_{A_n} \leq f 1_{A_n}$. So

$$\int f d\mu \geq \lim_{n \rightarrow \infty} \int f 1_{A_n} d\mu \geq \lim_{n \rightarrow \infty} \int \varphi 1_{A_n} d\mu = \lim_{n \rightarrow \infty} \sum_{i=1}^m a_i \mu(B_i \cap A_n) = \sum_{i=1}^m a_i \mu(B_i \cap A)$$

$$= \int \varphi 1_A d\mu \stackrel{\varphi \leq f}{=} \int \varphi d\mu. \text{ Take supremum over } \varphi \text{ to get } \int f d\mu \text{ on RHS } \quad \square$$