

Random variables and (maybe) Lebesgue integral

Recall $(\Sigma, \mathcal{G}) = \text{meas. sp.}$, Σ -valued RV is a $\mathcal{F}/\mathcal{G}$ -measurable map $X: \Omega \rightarrow \Sigma$
 For $\mathbb{R}$ -valued, use $\Sigma := \mathbb{R}$, $\mathcal{G} := \mathcal{B}(\mathbb{R})$.
 (st. $\text{Dom}(X) = \Omega$)

Lemma If X, Y are $\mathbb{R}$ -valued RV's, then so are $X+Y, X \cdot Y$.

Pf: $\{X+Y > a\} = \bigcup_{z \in \mathbb{Q}} \{X > z\} \cap \{Y+z > a\} \in \mathcal{F}$
 so $(X+Y)^{-1}((a, \infty)) \in \mathcal{F}$, since $\sigma(\{(a, \infty): a \in \mathbb{R}\}) = \mathcal{B}(\mathbb{R})$, $X+Y$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$ -meas.
 For $X \cdot Y$ proceed similarly $\square$

Cor $\{X \in \mathbb{R}^\Omega: \mathcal{F}/\mathcal{B}(\mathbb{R})\text{-meas}\}$ is a vector space

$1_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$
 indicator function

Lemma If X is a RV on $(\Omega, \mathcal{F})$ and $A \in \mathcal{F}$, then $X 1_A$ is a RV.

Pf $\{X 1_A > a\} \stackrel{a \geq 0}{=} A \cap \{X > a\}, \{X 1_A \leq a\} \stackrel{a < 0}{=} A \cap \{X \leq a\}$

Lemma If X is $\mathbb{R}$ -valued RV and $f: \mathbb{R} \rightarrow \mathbb{R}$ Borel function, then $f(X)$ is RV.
Pf $(f \circ X)^{-1}(B) = X^{-1}(f^{-1}(B)) \quad \square$

E.g. $X^2, X^3, X^4, \dots, X^n$
 polynomials in X
 $e^{\lambda X}, \sin(\lambda X), \cos(\lambda X), \dots$

Behavior under pointwise limits: Need to work with $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$
 with topology generated by $\{(a, b) : a < b\} \cup \{(-\infty, a), (a, \infty) : a \in \mathbb{R}\} \cup \{\bar{\mathbb{R}}\}$

Lemma If $\{X_n\}_{n \in \mathbb{N}}$ are $\bar{\mathbb{R}}$ -valued RV's, then so are
 $\sup_{n \in \mathbb{N}} X_n$, $\inf_{n \in \mathbb{N}} X_n$, $\limsup_{n \rightarrow \infty} X_n$, $\liminf_{n \rightarrow \infty} X_n$

Pf $\{\sup_{n \in \mathbb{N}} X_n > a\} = \bigcup_{n \in \mathbb{N}} \{X_n > a\}$, etc.

Lemma Let $\{X_n\}_{n \in \mathbb{N}}$ be $\mathbb{R}$ -valued RV's, on $(\Omega, \mathcal{F})$. Then

- 1) $\{\lim_{n \rightarrow \infty} X_n \text{ exists in } \mathbb{R}\} \in \mathcal{F}$
- 2) Setting, for any $a \in \mathbb{R}$, $X := \begin{cases} \lim_{n \rightarrow \infty} X_n & \text{on } \{\lim_{n \rightarrow \infty} X_n \text{ exists in } \mathbb{R}\} \\ a & \text{on } \{\lim_{n \rightarrow \infty} X_n \text{ does not exist}\} \end{cases}$
 gives us a RV X .

Pf $\{\lim_{n \rightarrow \infty} X_n \text{ exists in } \bar{\mathbb{R}}\} = \left(\bigcup_{q \in \mathbb{Q}} \{\liminf_{n \rightarrow \infty} X_n < q < \limsup_{n \rightarrow \infty} X_n\} \right)^c$
 $\{\limsup_{n \rightarrow \infty} X_n < \infty\} = \bigcup_{q \in \mathbb{Q}} \{\limsup_{n \rightarrow \infty} X_n < q\}$

Limits allow for approximations

Key object.

Def A RV X is simple if $\text{Ran}(X)$ is finite, or
Equivalently, if $\exists n \geq 1 \exists a_1, \dots, a_n \in \mathbb{R} \exists A_1, \dots, A_n \in \mathcal{F}: X = \sum_{i=1}^n a_i 1_{A_i}$

Lemma Let X be $\mathbb{R}$ -valued RV. Then $\exists \{X_n\}_{n \in \mathbb{N}}$ of RV's s.t.

$\forall n \in \mathbb{N}: X_n$ is simple and $X_n \rightarrow X$ (pointwise) ($\forall \omega \in \Omega: X(\omega) = \lim_{n \rightarrow \infty} X_n(\omega)$)

If $X \geq 0$, we can choose $\{X_n\}$ s.t. $\forall n \in \mathbb{N}: 0 \leq X_n \leq X_{n+1}$ and so $X_n \uparrow X$.

Pf: $X = X^+ - X^-$ where $X^+ := \max\{X, 0\}$, $X^- := \max\{-X, 0\}$.

So assume $X \geq 0$. Set $X_n := \sum_{k=0}^{4^n-1} k 2^{-n} 1_{\{k 2^{-n} \leq X < (k+1) 2^{-n}\}}$.

Note $\{k 2^{-n} \leq X < (k+1) 2^{-n}\}$
 $= \{2k 2^{-(n+1)} \leq X < (2k+1) 2^{-(n+1)}\} \cup \{(2k+1) 2^{-(n+1)} \leq X < (2k+2) 2^{-(n+1)}\}$

$X_{n+1} = 2k 2^{-(n+1)} = k 2^{-n} = X_n$ on first event | so $X_{n+1} \geq X_n$

$X_{n+1} = (2k+1) 2^{-(n+1)} > k 2^{-n} = X_n$ on 2nd event |

Note on $\{X \leq 2^{-n}\}$
we have $|X_n - X| \leq 2^{-n}$

Note $\Omega = \bigcup_{n \geq 0} \{X \leq 2^{-n}\}^c$ $\square$

Thm (Monotone class theorem) Let $\mathcal{M}_+ := \{X \in \mathbb{R}^\Omega : \mathcal{F}/\mathcal{B}(\mathbb{R})\text{-meas.}, X \geq 0\}$

Suppose $\mathcal{M}' \subseteq \mathcal{M}_+$ is s.t.

- 1) $\mathcal{M}'$ contains all non-negative simple RV's.
- 2) $\mathcal{M}'$ is closed under upward monotone pointwise limits.

Then $\mathcal{M}' = \mathcal{M}_+$.

Corollary Given RV, set $\sigma(X) := \{ \{X \in B\} : B \in \mathcal{B}(\mathbb{R}) \}$.

Then for any $\sigma(X)/\mathcal{B}(\mathbb{R})$ -meas. $Y : \Omega \rightarrow \mathbb{R}$, there exists Borel $f : \mathbb{R} \rightarrow \mathbb{R}$ s.t. $Y = f(X)$.

$$\{X \in B\} = X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$$

Pf: $\mathcal{M}_+ = \{Y \in \mathbb{R}_+^\Omega : \sigma(X)/\mathcal{B}(\mathbb{R})\text{-meas.}\}$, $\mathcal{M}' = \{f(X) : f \in \mathbb{R}_+^{\mathbb{R}} \text{ Borel}\}$.

$\mathcal{M}'$ contains simple RV: Y simple, $\sigma(X)/\mathcal{B}(\mathbb{R})$ -meas. Then $Y = \sum_{i=1}^n a_i 1_{A_i}$ wlog $a_1, \dots, a_n$ distinct, $A_1, \dots, A_n$ disjoint.

Then $A_i = \{Y = a_i\} = \{X \in B_i\}$ for some $B_i \in \mathcal{B}(\mathbb{R})$. Let $f(x) = \sum_{i=1}^n a_i 1_{B_i}(x)$ and $f(X) = \sum_{i=1}^n a_i 1_{X \in B_i} = Y$.

M' closed under $\uparrow$ limits. Let $\{f_n\}$ Borel functions s.t. $\{f_n(x)\}$ is $\uparrow$.
Define $f(x) := \sup_{n \in \mathbb{N}} f_n(x)$. Then $f_n(x) \uparrow f(x)$ so $f(x) \in M'$.

So $M' = M_+$. $\square$

On the set where supremum = infinity, set $f(x) := 0$ instead

Note $\Omega = \bigcup_{n \geq 0} \{X \leq 2^{-n}\}$ $\square$