

Finishing measure (for now) & moving to functions & RV

recall: • μ_1, μ_2 finite, agree on π -system $\mathcal{C} \Rightarrow$ agree on $\sigma(\mathcal{C})$.
 • if μ_1, μ_2 infinite, we need:

Def μ on $(\Omega, \mathcal{F})$ is called σ -finite if $\exists \{\Omega_n\}_{n \in \mathbb{N}} \in \mathcal{F}^{\mathbb{N}}$:
 $\forall n \in \mathbb{N}: \mu(\Omega_n) < \infty \wedge \Omega_n \uparrow \Omega$.

if μ_1, μ_2 are σ -finite, then equality on π -system $\mathcal{C} \Rightarrow$ equality on $\sigma(\mathcal{C})$.

$\mu_F(\{b\}) = F(b) - F(b^-)$
 so point masses $\Leftrightarrow$ discontinuities of F

As a consequence: $F \leftrightarrow \mu_F$ given by $\mu_F((a, b]) = F(b) - F(a)$ is one-to-one.
 $(\mathbb{R}, \mathcal{B})$

Ex: • $F(x) := \begin{cases} 1 & x \geq a \\ 0 & x < a \end{cases}$

• $F(x) := x$

• $F(x) := \lfloor x \rfloor$

• $F \in C^1(\mathbb{R}) (\uparrow, \mathbb{R})$

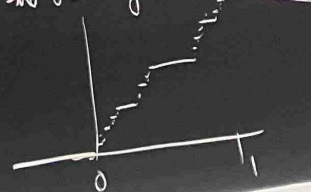
• $F(x) = \sum_{i=1}^{\infty} \frac{2^i x_i}{3^i}$ when $x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}$ ($x_i \in \{0, 1\}$)
 $(F(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 1 \end{cases})$ $\mu_F =$ "uniform" measure on Cantor set.

$\mu_F = \delta_a$
 $\mu_F((a, b]) = b - a$ Lebesgue measure (on $\mathcal{B}(\mathbb{R})$)

$\mu_F =$ counting measure on $\mathbb{Z}$.

$\mu_F((a, b]) = \int_a^b F'(s) ds$

F' is the density w.r.t. Lebesgue measure



Q: In the construction of μ_F , we constructed B as μ^* on $\mathcal{F}(\mu^*)$ and restricted to $\sigma(\mathcal{C})$ covering class.
How much bigger is $\mathcal{F}(\mu^*)$ from $\sigma(\mathcal{C})$?

Lemma $\forall A \in \mathcal{F}(\mu^*)$: $\mu^*(A) = 0 \Rightarrow (\forall B \subseteq A: B \in \mathcal{F}(\mu^*) \wedge \mu^*(B) = 0)$

Pf: Let $E \subseteq \Omega$ (wlog $\mu^*(E) < \infty$), $A \in \mathcal{F}(\mu^*)$ with $\mu^*(A) = 0$, let $B \subseteq A$.

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) = \mu^*(E \cap A^c) \\ &\stackrel{\text{sigma-finite}}{\geq} \mu^*(E \cap B^c) - \mu^*(A \setminus B) = \mu^*(E \cap B^c) = \mu^*(E \cap B) + \mu^*(E \cap B^c) \geq \mu^*(E) \end{aligned}$$

So $B \in \mathcal{F}(\mu^*)$ and $\mu^*(B) = 0$. $\square$

Def $(\Omega, \mathcal{F}, \mu)$ is complete if $\forall A \in \mathcal{F} \forall B \subseteq A: \mu(A) = 0 \Rightarrow B \in \mathcal{F} \wedge \mu(B) = 0$.

Lemma Let $\mathcal{C}$ be the covering class and assume $\mathcal{C} \subseteq \mathcal{F}(\mu^*)$. Then $\forall A \in \mathcal{F}(\mu^*)$ with $\mu^*(A) < \infty$:

$$1) \forall \varepsilon > 0 \exists B \in \mathcal{C}_\sigma: A \subseteq B \wedge \mu^*(B \setminus A) < \varepsilon$$

$$2) \exists C \in \mathcal{C}_{\sigma\delta}: A \subseteq C \wedge \mu^*(C \setminus A) = 0$$

In particular, if $\mu^*(\Omega) < \infty$, then

$$\mathcal{F}(\mu^*) = \{C \setminus N: C \in \mathcal{C}_{\sigma\delta}, \mu^*(N) = 0 \wedge N \subseteq C\}$$

$$\mathcal{C}_\sigma := \left\{ \bigcup_{i \in \mathbb{N}} B_n: \{B_n\}_{n \in \mathbb{N}} \in \mathcal{C}^{\mathbb{N}} \right\}$$

$$\mathcal{C}_\delta := \left\{ \bigcap_{i \in \mathbb{N}} B_n: \text{---} \text{---} \text{---} \right\}$$

Pf 1) $\mu^*(A) < \infty \Rightarrow \forall \varepsilon > 0 \exists \{B_n\} \in \mathcal{C}^N : A \subseteq \bigcup_{n \in \mathbb{N}} B_n \wedge$
 $\mu^*(A) + \varepsilon \geq \sum_{n \in \mathbb{N}} \mu(B_n) \geq \sum_{n \in \mathbb{N}} \mu^*(B_n) \geq \mu^*\left(\bigcup_{n \in \mathbb{N}} B_n\right)$
 Set $B := \bigcup_{n \in \mathbb{N}} B_n$. Then $C \subseteq \mathcal{F}(\mu^*) \Rightarrow B \in \mathcal{C}_\sigma \subseteq \mathcal{F}(\mu^*)$
 Since μ^* is a measure we get $\mu^*(A) + \varepsilon \geq \mu^*(B) = \mu^*(A) + \mu^*(B \setminus A)$
 Subtract $\mu^*(A)$ on both sides.

2) By 1) $\forall n \geq 0 \exists C_n \in \mathcal{C}_\sigma$ s.t. $A \subseteq C_n \wedge \mu^*(C_n \setminus A) < \frac{1}{n}$.
 Hence, $\mu^*\left(\left(\bigcap_{k=1}^{\infty} C_k\right) \setminus A\right) \leq \mu^*(C_n \setminus A) < \frac{1}{n} \xrightarrow{n \rightarrow \infty} 0$
 Set $C := \bigcap_{k=1}^{\infty} C_k$, we get $\mu^*(C \setminus A) = 0$. Note $C \in \mathcal{C}_{\sigma\delta}$.

Corollary: $\sigma(\mathcal{C}) = \{C \setminus N : C \in \mathcal{C}_{\sigma\delta} \wedge N \in \sigma(\mathcal{C}) \wedge \mu^*(N) = 0\}$.

Ex • $F(x) := \begin{cases} 1 & x \geq a \\ 0 & x < a \end{cases}$

• $F(x) := x$

$\mathcal{F}(\mu^*) = \mathcal{P}(\mathbb{R})$

$\mathcal{F}(\mu^*) = \mathcal{L}(\mathbb{R}) := \text{Lebesgue meas. sets.}$

Thm (Vitali) Assuming AC: $\mathcal{L}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$

Note $\mathcal{B}(\mathbb{R}) \neq \mathcal{L}(\mathbb{R})$

Back to RV's:

Recall $(\Omega, \mathcal{F}), (\Sigma, \mathcal{G}) = \text{measurable space}$,

$f: \Omega \rightarrow \Sigma$ is $\mathcal{F}/\mathcal{G}$ -measurable if $\forall B \in \mathcal{G}: f^{-1}(B) \in \mathcal{F}$ ($f^{-1}(\mathcal{G}) \subseteq \mathcal{F}$)

Q: How do we check measurability?

Lemma (Helper) If $f: \Omega \rightarrow \Sigma$, then

- 1) $\{B \subseteq \Sigma: f^{-1}(B) \in \mathcal{F}\}$ is σ -alg if $\mathcal{F}$ is σ -alg.
- 2) $\{f^{-1}(B): B \in \mathcal{G}\}$ is σ -alg if $\mathcal{G}$ is σ -alg.

Corollary: Let $\mathcal{C} \subseteq \mathcal{G}$ be s.t. $\sigma(\mathcal{C}) = \mathcal{G}$. Then (for $\mathcal{F}$ - σ -alg on Ω , $f: \Omega \rightarrow \Sigma$)

$f^{-1}(\mathcal{C}) \subseteq \mathcal{F} \Rightarrow f^{-1}(\mathcal{G}) \subseteq \mathcal{F}$ and so f is $\mathcal{F}/\mathcal{G}$ -meas.

Pf: $\tilde{\mathcal{G}} := \{B \subseteq \Sigma: f^{-1}(B) \in \mathcal{F}\}$, $\mathcal{F}$ σ -alg $\Rightarrow \tilde{\mathcal{G}}$ σ -alg. $f^{-1}(\mathcal{C}) \subseteq \mathcal{F} \Rightarrow \mathcal{C} \subseteq \tilde{\mathcal{G}} \Rightarrow \sigma(\mathcal{C}) \subseteq \tilde{\mathcal{G}} \quad \square$

Corollary: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Then f is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ -meas.

Lemma $\mathcal{L}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R}) \Rightarrow \exists f: \mathbb{R} \rightarrow \mathbb{R}$ continuous but NOT $\mathcal{L}(\mathbb{R})/\mathcal{L}(\mathbb{R})$ -measurable.