

Measure extension: uniqueness

Lemma: Let $F: \mathbb{R} \rightarrow \mathbb{R}$ be right-continuous and non-decreasing. (Dom $F = \mathbb{R}$)
Then there exists a measure μ_F on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ s.t.

$$\forall a < b: \mu_F((a, b]) = F(b) - F(a)$$

Pf $\mathcal{P} := \{\emptyset, \mathbb{R}\} \cup \{(a, b]: -\infty \leq a < b \leq +\infty\}$ semialgebra
 $\mu_0((a, b]) = F(b) - F(a)$. NTS μ_0 is finitely add. & countably subadd. on $\mathcal{P}$

$$\text{Claim 1: } (a, b] = \bigcup_{i=1}^n (a, b_i] \Rightarrow F(b) - F(a) = \sum_{i=1}^n [F(b_i) - F(a)]$$

Pf. Order s.t. $a_1 < a_2 < \dots < a_n$. Then $b_i = a_{i+1}$ for $i=1, \dots, n-1$.

$$\text{Now } F(b) - F(a) = F(a_n) - F(a_1) = \sum_{i=1}^{n-1} [F(a_{i+1}) - F(a_i)] = \sum_{i=1}^{n-1} [F(b_i) - F(a_i)]$$

$$\text{Claim 2: } (a, b] \subseteq \bigcup_{i=1}^n (a_i, b_i] \Rightarrow F(b) - F(a) \leq \sum_{i=1}^n [F(b_i) - F(a_i)]$$

Pf: Label $\{a_i, b_i\} \cup \{a, b\}$ into $c_1 < c_2 < \dots < c_m$
 $(a_i, b_i] = \bigcup_{l=j_i}^{k_i} (c_l, c_{l+1}]$

Then $\forall i=1, \dots, n \exists 1 \leq j_i < k_i \leq m$
also $\exists 1 \leq j < k \leq m$ s.t. $(a, b] = \bigcup_{l=j}^{k-1} (c_l, c_{l+1}]$

Now: $F(b) - F(a) \stackrel{\text{claim 1}}{=} \sum_{j=1}^{k-1} [F(c_{j+1}) - F(c_j)] \leq \sum_{i=1}^n \sum_{j=i}^{k-1} \underbrace{[F(c_{j+1}) - F(c_j)]}_{\geq 0} \stackrel{\text{claim 1}}{=} \sum_{i=1}^n [F(b_i) - F(a_i)]$

claim 3 $(a, b] = \bigcup_{i \in \mathbb{N}} (a, b_i] \Rightarrow F(b) - F(a) \leq \sum_{i \in \mathbb{N}} [F(b_i) - F(a_i)]$

Pf. Pick $\varepsilon > 0$, use RC of F to find:
 $\exists a' \in (a, b) : F(a') \leq F(a) + \varepsilon$ and $\forall i \in \mathbb{N} \exists b'_i > b_i : F(b'_i) \leq F(b_i) + \varepsilon 2^{-i}$

Now: $[a', b] \subseteq (a, b] \subseteq \bigcup_{i \in \mathbb{N}} (a, b_i] \subseteq \bigcup_{i \in \mathbb{N}} (a, b'_i)$

Heine-Borel: $\exists N \in \mathbb{N} : [a', b] \subseteq [a, b] \subseteq \bigcup_{i=1}^N (a, b'_i) \subseteq \bigcup_{i=1}^N (a, b_i]$
↑
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by claim 2: $F(b) - F(a) - \varepsilon \leq F(b) - F(a') \stackrel{\text{claim 2}}{\leq} \sum_{i=1}^N [F(b'_i) - F(a_i)]$

Now take $\varepsilon \downarrow 0$ to get the claim.

$= \sum_{i=1}^N [F(b_i) - F(a_i) + \varepsilon 2^{-i}] \leq \sum_{i \in \mathbb{N}} [F(b_i) - F(a_i)] + \varepsilon$

So we showed that $(\mathcal{F}, \mu_0)$ obey the needed conditions.

Lemma + Hahn-Kolmogoroff theorem: μ_0 extends to $\mathcal{F}(\mu^*)$, and $\mathcal{F} \subseteq \mathcal{F}(\mu^*)$.

Now observe: $(a, b] = \bigcup_{n \in \mathbb{N}} (a, b - 1/n]$, every open set in $\mathbb{R}$ is thus a member of $\sigma(\mathcal{F})$. Hence, $\mathcal{B}(\mathbb{R}) \subseteq \sigma(\mathcal{F}) \subseteq \mathcal{F}(\mu^*)$



Key remaining issue: uniqueness.

Lemma Let μ_1, μ_2 be finite measures on $(\Omega, \mathcal{F})$ s.t. $\mu_1(\Omega) = \mu_2(\Omega)$.
Set $\mathcal{L} := \{A \in \mathcal{F} : \mu_1(A) = \mu_2(A)\}$ (same "mass")

Then

- 1) $\emptyset, \Omega \in \mathcal{L}$
- 2) $\forall A, B \in \mathcal{L} : A \subseteq B \Rightarrow B \setminus A \in \mathcal{L}$
- 3) $\forall \{A_n\}_{n \in \mathbb{N}} \in \mathcal{L}^{\mathbb{N}} \quad \forall A \in \mathcal{P}(\Omega) : A_n \uparrow A \Rightarrow A \in \mathcal{L}$.

(Note: $A \in \mathcal{L} \Leftrightarrow A^c \in \mathcal{L}$
 $A_n \downarrow A \Rightarrow A \in \mathcal{L}$)

Pf By properties of measures. Finiteness key for (2).

Note $\mathcal{L}$ need not be an algebra!

Ex $\Omega = \{1, 2, 3, 4\}$, $\mathcal{F} = \mathcal{P}(\Omega)$
 $\mu_1(\{k\}) = \begin{cases} a & k=1 \\ b & k=2 \\ b & k=3 \\ a & k=4 \end{cases}$ $\mu_2(\{k\}) = \begin{cases} a & k=1 \\ a & k=2 \\ b & k=3 \\ b & k=4 \end{cases}$
s.t. $a+b = 1/2 = \tilde{a} + \tilde{b}$ yet $a \neq \tilde{a}$
then $\{1, 2\} \in \mathcal{L}$, $\{1, 3\} \in \mathcal{L}$ yet $\{1, 2\} \cap \{1, 3\} = \{1\} \notin \mathcal{L}$.

Def Any $\mathcal{L} \subseteq \mathcal{P}(\Omega)$ satisfying (1-3) is called λ -system

Def Any $\mathcal{P} \subseteq \mathcal{P}(\Omega)$ s.t. $\forall A, B \in \mathcal{P} : A \cap B \in \mathcal{P}$ is called π -system

Thm (Dynkin's π/λ -theorem) Let $\mathcal{L} \subseteq \mathcal{P}(\Omega)$ be λ -system and $\mathcal{P} \subseteq \mathcal{P}(\Omega)$ a π -system.

Thm $\mathcal{P} \subseteq \mathcal{L} \Rightarrow \sigma(\mathcal{P}) \subseteq \mathcal{L}$.

Pf: Given $\mathcal{P} \subseteq \mathcal{L}$, let $\mathcal{L}(\mathcal{P}) = \bigcap \{ \mathcal{L}' : \lambda\text{-system} \wedge \mathcal{P} \subseteq \mathcal{L}' \}$. Will show the claim with $\mathcal{L}(\mathcal{P})$ replacing $\mathcal{L}$. Denote $\mathcal{L}_A := \{ B \in \mathcal{L}(\mathcal{P}) : A \cap B \in \mathcal{L}(\mathcal{P}) \} \stackrel{?}{=} \lambda\text{-system}$.

Now: $\mathcal{P} \subseteq \mathcal{L}(\mathcal{P}) \stackrel{\pi\text{-system}}{\Rightarrow} \forall A \in \mathcal{P} : \mathcal{P} \subseteq \mathcal{L}_A$
 $\Rightarrow \forall A \in \mathcal{P} : \mathcal{L}(\mathcal{P}) \subseteq \mathcal{L}_A$
 $\Leftrightarrow \forall A \in \mathcal{P} \forall B \in \mathcal{L}(\mathcal{P}) : A \cap B \in \mathcal{L}(\mathcal{P})$
 $\Rightarrow \forall B \in \mathcal{L}(\mathcal{P}) : \mathcal{P} \subseteq \mathcal{L}_B$

$$\left| \sigma(A) = \bigcap \{ \mathcal{G} : \sigma\text{-alg} \atop A \in \mathcal{G} \} \right|$$

$\Rightarrow \forall B \in \mathcal{L}(\mathcal{P}) : \mathcal{L}(\mathcal{P}) \subseteq \mathcal{L}_B$
 $\Leftrightarrow \forall A \in \mathcal{L}(\mathcal{P}) \forall B \in \mathcal{L}(\mathcal{P}) : A \cap B \in \mathcal{L}(\mathcal{P})$.

Since $\mathcal{L}(\mathcal{P})$ is λ -system closed under $\cap \Rightarrow \mathcal{L}(\mathcal{P})$ is σ -alg. Hence: $\sigma(\mathcal{P}) \subseteq \mathcal{L}(\mathcal{P})$. $\square$

Corollary: Let μ_1, μ_2 be two measures on $(\Omega, \mathcal{F})$ s.t. $\exists \{ \Omega_n \}_{n \in \mathbb{N}} \in \mathcal{F}^{\mathbb{N}}$ with $\Omega_n \uparrow \Omega$ and $\forall n \in \mathbb{N} : \mu_1(\Omega_n) = \mu_2(\Omega_n) < \infty$.
 If $\mathcal{P} \subseteq \mathcal{F}$ is π -system s.t. $\sigma(\mathcal{P}) = \mathcal{F}$ and $\forall n \in \mathbb{N} : \Omega_n \in \mathcal{P}$, then

$$\forall A \in \mathcal{P} : \mu_1(A) = \mu_2(A) \Rightarrow \forall A \in \mathcal{F} : \mu_1(A) = \mu_2(A)$$