

Measure extension completed

last time: $\mathcal{C}$ = covering class, $\mu_0: \mathcal{C} \rightarrow [0, \infty]$ premeasure $\mapsto \mu^*$ outer measure
Carathéodory. $\mathcal{F}(\mu^*) := \{A \subseteq \Omega: (\forall E \subseteq \Omega: \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c))\}$
is σ -alg and $\mu^*|_{\mathcal{F}(\mu^*)}$ is a measure

Q: When do we have $\mathcal{C} \subseteq \mathcal{F}(\mu^*)$ and $\mu^*|_{\mathcal{C}} = \mu_0$?

A: Suffices that μ_0 is free of inconsistencies (with σ -additivity)

$$\mu^*(A) = \inf \left\{ \sum_{i \in \mathbb{N}} \mu_0(B_i) : \{B_i\}_{i \in \mathbb{N}} \subset \mathcal{C} \right. \\ \left. \wedge A \subseteq \bigcup_{i \in \mathbb{N}} B_i \right\}$$

Thm (Hahn-Kolmogorov) For $\mathcal{C}, \mu_0$ and μ^* as above, if $\mathcal{C}$ is an algebra and μ_0 is

1) (finitely additive on $\mathcal{C}$):

$$\forall A, B \in \mathcal{C}: A \cap B = \emptyset \Rightarrow \mu_0(A \cup B) = \mu_0(A) + \mu_0(B)$$

2) (countably subadd. on $\mathcal{C}$):

$$\forall \{A_n\}_{n \in \mathbb{N}} \in \mathcal{C}^{\mathbb{N}}: \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{C} \Rightarrow \mu_0\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu_0(A_n)$$

Then: $\mathcal{C} \subseteq \mathcal{F}(\mu^*) \wedge \forall A \in \mathcal{C}: \mu^*(A) = \mu_0(A)$.

• 2) Suffices to check for disjoint unions.
• $\leq$ is the monotonic direction in 2
as 1) $\Rightarrow \geq$ in 2).

Pf: Let $A \in \mathcal{C}$, $E \subseteq \Omega$. Since only $\geq$ needs to be shown in def of $\mathcal{F}(\mu^*)$, wlog: $\mu^*(E) < \infty$.
 Given $\varepsilon > 0$, there are $\{B_i\}_{i \in \mathbb{N}} \in \mathcal{C}^{\mathbb{N}}$ s.t. $E \subseteq \bigcup_{i \in \mathbb{N}} B_i$ and $\mu^*(E) + \varepsilon \geq \sum_{i \in \mathbb{N}} \mu_0(B_i)$

$$\text{Now } \mu^*(E) + \varepsilon \geq \sum_{i \in \mathbb{N}} \mu_0(B_i) \stackrel{(1)}{=} \sum_{i \in \mathbb{N}} (\mu_0(B_i \cap A) + \mu_0(B_i \cap A^c))$$

$$\stackrel{\text{def } \mu^*}{\geq} \mu^*\left(\bigcup_{i \in \mathbb{N}} B_i \cap A\right) + \mu^*\left(\bigcup_{i \in \mathbb{N}} B_i \cap A^c\right) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c) \geq \mu^*(E)$$

So taking $\varepsilon \downarrow 0$ we get $A \in \mathcal{F}(\mu^*)$.

Next, let $A \in \mathcal{C}$. Then $A \subseteq A \cup \bigcup_{i \in \mathbb{N}} \emptyset$ s.t. $\mu^*(A) \leq \mu_0(A)$. (wlog $\mu^*(A) < \infty$)

For $\geq$ let $\{B_i\}_{i \in \mathbb{N}} \in \mathcal{C}^{\mathbb{N}}$ be s.t. $A \subseteq \bigcup_{i \in \mathbb{N}} B_i$. Then

$$\sum_{i \in \mathbb{N}} \mu_0(B_i) \geq \sum_{i \in \mathbb{N}} \mu_0(B_i \cap A) \stackrel{(2)}{\geq} \mu_0\left(\bigcup_{i \in \mathbb{N}} \underbrace{B_i \cap A}_{= A \in \mathcal{C}}\right) = \mu_0(A) \Rightarrow \mu^*(A) \geq \mu_0(A) \quad \square$$

Def A semialgebra is a set $\mathcal{F} \subseteq \Omega$ s.t.

1) $\emptyset, \Omega \in \mathcal{F}$

2) $\forall A, B \in \mathcal{F}: A \cap B \in \mathcal{F}$

3) $\forall A \in \mathcal{F} \exists \{B_i\}_{i=1}^{\infty} \in \mathcal{F}^{\mathbb{N}}: \{B_i\} \text{ disjoint and } A^c = \bigcup_{i=1}^{\infty} B_i$

Ex $\mathcal{F}_1 = \{\emptyset, \mathbb{R}\} \cup \{(a, b] : -\infty \leq a < b \leq +\infty\}$

$\mathcal{F}_2 = \{\emptyset, \mathbb{R}^d\} \cup \left\{ \bigcap_{i=1}^d (a_i, b_i] : -\infty \leq a_i < b_i \leq +\infty \right\}$



Lemma Let $\mathcal{F}$ be a σ -algebra on Ω . Set

$$A := \left\{ \bigcup_{i=1}^n A_i : n \geq 1 \wedge \{A_i\}_{i=1}^n \in \mathcal{F}^n \wedge \{A_i\}_{i=1}^n \text{ disjoint} \right\}$$

 Then A is an algebra. Moreover, if $\mu_0: \mathcal{F} \rightarrow [0, \infty]$ is s.t.

1) μ_0 is finitely add. on $\mathcal{F}$:

$$\forall A, B \in \mathcal{F}: A \cap B = \emptyset \wedge A \cup B \in \mathcal{F} \Rightarrow \mu_0(A \cup B) = \mu_0(A) + \mu_0(B)$$

2) μ_0 is countably subadd. on $\mathcal{F}$:

$$\forall \{A_i\}_{i \in \mathbb{N}} \in \mathcal{F}^{\mathbb{N}}: \{A_i\}_{i \in \mathbb{N}} \text{ disjoint} \wedge \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F} \Rightarrow \mu_0\left(\bigcup_{i \in \mathbb{N}} A_i\right) \leq \sum_{i \in \mathbb{N}} \mu_0(A_i)$$

Then $\forall \{A_i\}_{i=1}^n \in \mathcal{F}^n \forall \{B_j\}_{j=1}^m \in \mathcal{F}^m: \text{disjoint} \wedge \bigcup_{i=1}^n A_i = \bigcup_{j=1}^m B_j \Rightarrow \sum_{i=1}^n \mu_0(A_i) = \sum_{j=1}^m \mu_0(B_j)$.

and, defining $\bar{\mu}_0: A \rightarrow [0, \infty]$ by

$$\bar{\mu}_0\left(\bigcup_{i=1}^n A_i\right) := \sum_{i=1}^n \mu_0(A_i)$$

we get that

$\bar{\mu}_0$ is finitely additive and countably subadditive on A .

Pf. HW 2

Note $\bar{\mu}_0$ then extends to a measure on $\mathcal{F}(\mu_0^*)$ which contains

$$\sigma(\mathcal{F}) = \bigcap \left\{ \mathcal{G} \subseteq \mathcal{P}(\Omega) : \sigma\text{-alg} \wedge \mathcal{G} \supseteq \mathcal{F} \right\}$$

Application: Stieljes measures on $\mathbb{R}$.

- Given $F: \mathbb{R} \rightarrow \mathbb{R}$, want a measure on $\mathcal{B}(\mathbb{R})$ s.t. $\forall a < b: \mu((a, b]) = F(b) - F(a)$
- in prob., this means finding a distribution of a RV s.t. $\forall x \in \mathbb{R}: P(X \leq x) = F(x)$ CDF

Thm Let $F: \mathbb{R} \rightarrow \mathbb{R}$ be non-decreasing, right continuous. Then there exists a measure μ_F on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ s.t.

$$\forall -\infty \leq a < b \leq +\infty: \mu((a, b]) = F(b) - F(a)$$

$$F(+\infty) = \lim_{b \rightarrow +\infty} F(b)$$

$$F(-\infty) = \lim_{a \rightarrow -\infty} F(a)$$

Q: Why RC? $\bigcap_{n \geq 1} \{X \leq x + \frac{1}{n}\} = \{X \leq x\}$

Pf Define μ_0 on $\mathcal{Y} := \{\emptyset, \mathbb{R}\} \cup \{(a, b]: -\infty \leq a < b \leq +\infty\}$ by $\mu_0((a, b]) = F(b) - F(a)$
claim: μ_0 is finitely additive:

Suppose $(a, b] = \bigcup_{i=1}^n (a_i, b_i]$ wlog. assume $a_1 < b_1 = a_2 < b_2 < \dots < a_n < b_n =: a_{n+1}$

$$\text{Then } F(b) - F(a) = F(a_{n+1}) - F(a_1) = \sum_{k=1}^n (F(a_{k+1}) - F(a_k)) = \sum_{k=1}^n (F(b_k) - F(a_k))$$

Note if $(a, b] \subseteq \bigcup_{i=1}^n (a_i, b_i]$, then $F(b) - F(a) \leq \sum_{i=1}^n [F(b_i) - F(a_i)]$

Claim - no countably add on $\mathcal{F}$

Assume $(a, b] = \bigcup_{i \in \mathbb{N}} (a, b_i]$?

Given $\varepsilon > 0$, let $b'_i \in (b_i, b_i + \varepsilon)$ be s.t. $F(b'_i) \leq F(b_i) + \varepsilon 2^{-i}$
let $a' \in (a, b)$ s.t. $F(a') \leq F(a) + \varepsilon$

Now $[a', b] \subseteq \bigcup_{i \in \mathbb{N}} (a, b'_i)$

Heine-Borel: $\exists N \in \mathbb{N}$ s.t. $[a', b] \subseteq \bigcup_{i=1}^N (a, b'_i)$

Now: $F(b) - F(a) - \varepsilon \leq F(b) - F(a') \leq \sum_{i=1}^N [F(b'_i) - F(a)] \leq \sum_{i=1}^{\infty} (F(b) - F(a)) + \varepsilon$ TBC