

Measure extension problem

Issue: Measure (to be) prescribed on "small" class of sets.

Q: What conditions on setup ensure that the measure exists?

Related: Assign "length" to subsets of $\mathbb{R}$
"area" to subsets of $\mathbb{R}^2$



Idea Cover set from "above" by countable unions of good sets.

Lemma Let $\Omega = \text{set}$, $\mathcal{C} \subseteq \mathcal{P}(\Omega)$ s.t. $\emptyset \in \mathcal{C}$.

Given $\mu_0: \mathcal{C} \rightarrow [0, \infty]$ s.t. $\text{Dom}(\mu_0) = \mathcal{C} \wedge \mu_0(\emptyset) = 0$, set

$$\mu^*(A) := \inf \left\{ \sum_{i \in \mathbb{N}} \mu_0(C_i) : \{C_i\}_{i \in \mathbb{N}} \in \mathcal{C}^{\mathbb{N}} \wedge A \subseteq \bigcup_{i \in \mathbb{N}} C_i \right\}$$

$$\text{Then } \mu^*(\emptyset) = 0 \wedge \forall \{A_i\}_{i \in \mathbb{N}}: \mu^*\left(\bigcup_{i \in \mathbb{N}} A_i\right) \leq \sum_{i \in \mathbb{N}} \mu^*(A_i) \quad \text{☺}$$

P7. $\emptyset \subseteq \bigcup_{i \in \mathbb{N}} \emptyset$ so $0 \leq \mu^*(\emptyset) \leq 0$
 Let $\{A_i\}_{i \in \mathbb{N}} \in \mathcal{C}^{\mathbb{N}}$ be s.t. wlog $\forall i \in \mathbb{N}: \mu^*(A_i) < \infty$.
 Given 270, find $\{B_{n,i}\}_{i \in \mathbb{N}}$ s.t. $A_n \subseteq \bigcup_{i \in \mathbb{N}} B_{n,i} \wedge \mu^*(A_n) + \varepsilon 2^{-n} \geq \sum_{i \in \mathbb{N}} \mu_0(B_{n,i})$

Then $\bigcup_{n \in \mathbb{N}} A_n \subseteq \bigcup_{n \in \mathbb{N}} \bigcup_{i \in \mathbb{N}} B_{n,i}$ and so $\mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \sum_{i \in \mathbb{N}} \mu_0(B_{n,i}) \leq \sum_{n \in \mathbb{N}} (\mu^*(A_n) + \varepsilon 2^{-n})$
 $= \sum_{n \in \mathbb{N}} \mu^*(A_n) + \varepsilon$ □

Def Any μ^* satisfying ☺ is called an outer measure,

Note μ^* finitely subadditive
 μ^* is monotone

$$A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$$

Q: How do we select a subset of $\mathcal{P}(\Omega)$ s.t. μ^* restricted to it is a measure?
 (eg. $\{\emptyset, \Omega\}$ would work)
 So, need one s.t., hopefully, $\mathcal{C}$ is contained within.

Def (Carathéodory) We say $A \subseteq \Omega$ is Carathéodory measurable wrt. μ^* if
 $\forall E \subseteq \Omega: \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$

Notation: $\mathcal{M}(\mu^*) := \{A \subseteq \Omega; \text{Carathéodory meas. wrt. } \mu^*\}$.

Thm (Carathéodory) Let μ^* be an outer measure on Ω .

Then $\mathcal{F}(\mu^*)$ is σ -algebra $\wedge$ $\mu^*|_{\mathcal{F}(\mu^*)}$ is a measure

PF STEP 1: $\mathcal{F}(\mu^*)$ is an algebra

$\emptyset \in \mathcal{F}(\mu^*)$, $A \in \mathcal{F}(\mu^*) \Leftrightarrow A^c \in \mathcal{F}(\mu^*)$ by fiat

Let $A, B \in \mathcal{F}(\mu^*)$, $E \subseteq \Omega$, then

$$E \cap (A \cup B)^c = E \cap A^c \cap B^c \quad \wedge \quad E \cap (A \cup B) = E \cap A \cup (E \cap A^c \cap B)$$

So ...

$$\begin{aligned} & \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c) \\ & \stackrel{\text{subadd.}}{\leq} \mu^*(E \cap A^c \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A) \\ & \stackrel{\text{subadd.}}{\leq} \mu^*(E \cap A^c) + \mu^*(E \cap A) \stackrel{A, B \in \mathcal{F}(\mu^*)}{=} \mu^*(E) \leq \text{LHS} \end{aligned}$$

So $A \cup B \in \mathcal{F}(\mu^*)$ and $\mathcal{F}$ is an algebra. $\square$

STEP 2 $\forall \{A_i\}_{i=1}^n \in \mathcal{F}(\mu^*)^n \forall E \in \Omega: \{A_i\} \text{ disjoint} \Rightarrow \mu^*(E \cap \bigcup_{i=1}^n A_i) = \sum_{i=1}^n \mu^*(E \cap A_i)$
 In particular, μ^* is finitely additive on $\mathcal{F}(\mu^*)$

Pf by induction: $n=1$ triv, let $\{A_i\}_{i=1}^{n+1} \in \mathcal{F}(\mu^*)^{n+1}$ be disjoint. Then

$$\begin{aligned} \mu^*(E \cap \bigcup_{i=1}^{n+1} A_i) &\stackrel{A_n \in \mathcal{F}(\mu^*)}{=} \mu^*((E \cap \bigcup_{i=1}^n A_i) \cap A_{n+1}) + \mu^*((E \cap \bigcup_{i=1}^n A_i) \cap A_{n+1}^c) \\ &\stackrel{\text{disj.}}{=} \mu^*(E \cap A_{n+1}) + \mu^*(E \cap \bigcup_{i=1}^n A_i) \stackrel{\text{induct.}}{=} \mu^*(E \cap A_{n+1}) + \sum_{i=1}^n \mu^*(E \cap A_i) \quad \square \end{aligned}$$

STEP 3 $\mathcal{F}(\mu^*)$ is σ -algebra

Let $\{A_n\}_{n \in \mathbb{N}} \in \mathcal{F}(\mu^*)^{\mathbb{N}}$, set $A'_1 := A_1$ and $A'_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$ $n \geq 2$. Then $\{A'_i\}$ are disjoint and $\bigcup_{i \in \mathbb{N}} A_i = \bigcup_{i \in \mathbb{N}} A'_i$.
 $\mathcal{F}(\mu^*)$ being an algebra $\Rightarrow \forall i \in \mathbb{N}: A'_i \in \mathcal{F}(\mu^*)$.

Now for $E \in \Omega$:

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap \bigcup_{i=1}^{\infty} A'_i) + \mu^*(E \setminus \bigcup_{i=1}^{\infty} A'_i) \quad (\text{because } \bigcup_{i=1}^{\infty} A'_i \in \mathcal{F}(\mu^*)) \\ &\stackrel{\text{STEP 2}}{\geq} \sum_{i=1}^{\infty} \mu^*(E \cap A'_i) + \mu^*(E \setminus \bigcup_{i \in \mathbb{N}} A'_i) \xrightarrow{n \rightarrow \infty} \sum_{i \in \mathbb{N}} \mu^*(E \cap A'_i) + \mu^*(E \setminus \bigcup_{i \in \mathbb{N}} A'_i) \\ &\stackrel{\text{Subadd.}}{\geq} \mu^*(E \cap \bigcup_{i \in \mathbb{N}} A'_i) + \mu^*(E \setminus \bigcup_{i \in \mathbb{N}} A'_i) \geq \mu^*(E) \end{aligned}$$

So $\bigcup_{i \in \mathbb{N}} A_i = \bigcup_{i \in \mathbb{N}} A'_i \in \mathcal{F}(\mu^*)$. So $\mathcal{F}(\mu^*)$ is σ -alg. $\square$

STEP 4 μ^* is countably additive on $\mathcal{F}(\mu^*)$.

Let $\{A_n\}_{n \in \mathbb{N}} \in \mathcal{F}(\mu^*)$ be disjoint. Using argument from STEP 3 with $E := \bigcup_{i \in \mathbb{N}} A_i$ we get

$$\mu^*\left(\bigcup_{i \in \mathbb{N}} A_i\right) \geq \sum_{i \in \mathbb{N}} \mu^*(A_i) \stackrel{\text{sub.}}{\geq} \mu^*\left(\bigcup_{i \in \mathbb{N}} A_i\right)$$

So μ^* is a measure on $\mathcal{F}(\mu^*)$.



Q: When is $\mathcal{C} \subseteq \mathcal{F}(\mu^*)$ and $\mu^*|_{\mathcal{C}} = \mu_0$?

A: Certainly need to rule out inconsistencies.

This is the subject of Hahn-Kolmogorov's theorem.