

Towards measure theory

Last time: setup of prob $(\Omega, \mathcal{F}, P)$

we require $\mathcal{F}$ to conform to:

Def $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is σ -algebra if

1) $\emptyset, \Omega \in \mathcal{F}$

2) $\forall A \in \mathcal{F}: A^c \in \mathcal{F}$

3) $\forall \{A_n\}_{n \in \mathbb{N}} \in \mathcal{F}^{\mathbb{N}}: \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{F}$

From P we want:

Def: Given set Ω , σ -alg $\mathcal{F}$ on Ω , a measure is a map $\mu: \mathcal{F} \rightarrow [0, \infty]$

s.t. 1) $\mu(\emptyset) = 0$

2) $\forall \{A_n\}_{n \in \mathbb{N}} \in \mathcal{F}^{\mathbb{N}}: (\forall i, j \in \mathbb{N}: i \neq j \Rightarrow A_i \cap A_j = \emptyset) \Rightarrow \mu(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n)$

Def A measure P on $(\Omega, \mathcal{F})$ is a probability if $P(\Omega) = 1$ ↑ countable additivity

Note:
• $\mu(\emptyset) = 0$ not needed if $\mu(\Omega) < \infty$
• filling by $\emptyset \Rightarrow$ finite additivity

Thm Let μ be a meas. on $(\Omega, \mathcal{F})$. Then

- 1) $\forall A, B \in \mathcal{F}: A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$ (monotonicity)
- 2) $\forall \{A_n\}_{n \in \mathbb{N}} \in \mathcal{F}^{\mathbb{N}}: \mu(\bigcup_{n \in \mathbb{N}} A_n) \leq \sum_{n \in \mathbb{N}} \mu(A_n)$ (subadditivity, union bound)
- 3) $\forall \{A_n\}_{n \in \mathbb{N}} \in \mathcal{F}^{\mathbb{N}} \forall A \in \mathcal{F}: A_n \uparrow A \Rightarrow \mu(A_n) \uparrow \mu(A)$ (upward continuity)
- 4) $\forall \{A_n\}_{n \in \mathbb{N}} \in \mathcal{F}^{\mathbb{N}} \forall A \in \mathcal{F}: A_n \downarrow A \wedge \mu(A_1) < \infty \Rightarrow \mu(A_n) \downarrow \mu(A)$ (downward continuity)

Pr 1) $A \subseteq B \Rightarrow B = A \cup B \setminus A \Rightarrow \mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$

2) Define $A'_1 := A_1$ and $A'_n := A_n \setminus \bigcup_{j=1}^{n-1} A_j$ ($n \geq 2$). Then $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} A'_n$

Since A'_n 's are disjoint: $\mu(\bigcup_{n \in \mathbb{N}} A_n) = \mu(\bigcup_{n \in \mathbb{N}} A'_n) = \sum_{n \in \mathbb{N}} \mu(A'_n) \stackrel{(1)}{\leq} \sum_{n \in \mathbb{N}} \mu(A_n)$.

3) $A_n \uparrow A$ then $A = \bigcup_{j \in \mathbb{N}} (A_j \setminus A_{j-1})$ ($A_0 := \emptyset$)

and $A_n = \bigcup_{j=1}^n (A_j \setminus A_{j-1})$. So $\mu(A_n) = \sum_{j=1}^n \mu(A_j \setminus A_{j-1}) \xrightarrow{n \rightarrow \infty} \sum_{j=1}^{\infty} \mu(A_j \setminus A_{j-1}) = \mu(\bigcup_{j \in \mathbb{N}} (A_j \setminus A_{j-1})) = \mu(A)$

4) For $A_n \downarrow A$ set $B_n := A_1 \setminus A_n$, $B := A_1 \setminus A$. Then $B_n \uparrow B$.

So $\mu(A_n) = \mu(A_1) - \underbrace{\mu(A_1 \setminus A_n)}_{\substack{\uparrow \\ B_n}} \xrightarrow{\mu(A_1) < \infty} \mu(A_1) - \mu(A_1 \setminus A) = \mu(A)$ $\square$

Examples:

- Ω = finite or countably set

$$\mathcal{F} = \mathcal{P}(\Omega)$$

$$P(A) = \sum_{\omega \in A} f(\omega) \quad \text{where } f: \Omega \rightarrow [0,1] \text{ s.t. } \sum_{\omega \in \Omega} f(\omega) = 1$$

prob. mass. function

- Ω uncountable e.g. $\{0,1\}^{\mathbb{N}}$ or $[0,1]$

e.g. $\Omega = \{0,1\}^{\mathbb{N}}$ ∞ -sequence of coin tosses

$\mathcal{F} \supseteq$ events that depend on finite # of coordinates

$$P\left(\bigcap_{i=1}^n \{\omega_i = a_i\}\right) = \frac{1}{2^n} \quad \begin{matrix} \forall a_1, \dots, a_n \in \{0,1\} \\ \forall n \geq 1 \end{matrix}$$

Q: Do these identify $\mathcal{F}$ and P and do so uniquely?

or a "random" point in $[0,1]$?
call it X . want $P(X \in [a,b]) = (b-a) \quad \forall 0 \leq a < b \leq 1$.

$\mathcal{F}$ needs to contain $\{[a,b] : 0 \leq a < b \leq 1\}$
 P has to be defined on it.

Both lead to measure extension problem

X is an example of a random variable.

This a function $X: \Omega \rightarrow \mathbb{R}$ s.t. $\{X \in I\} = \{\omega \in \Omega : X(\omega) \in I\} = X^{-1}(I) \in \mathcal{F}$.

Lemma $\{B \subseteq \mathbb{R} : \{X \in B\} \in \mathcal{F}\}$ is σ -algebra.

Pf: $\{X \in \emptyset\} = \emptyset$, $\{X \in \mathbb{R}\} = \Omega$, $\{X \in B^c\} = \{X \in B\}^c$, $\{X \in \bigcup_{n \in \mathbb{N}} B_n\} = \bigcup_{n \in \mathbb{N}} \{X \in B_n\}$
Now use that $\mathcal{F}$ is σ -alg. $\square$

Note Intersection of any # of σ -alg. is σ -alg. so we take smallest containing all intervals.

Def Given $(\Omega, \mathcal{F})$ and another measurable space $(\Sigma, \mathcal{G})$
 a map $X: \Omega \rightarrow \Sigma$ is $\mathcal{F}/\mathcal{G}$ -measurable
 if $\forall B \in \mathcal{G}: \{X \in B\} \in \mathcal{F}$

Def A map $X: \Omega \rightarrow \mathbb{R}$ is $(\mathbb{R}$ -valued) ^{RV} random variable
 if X is $\mathcal{F}/\mathcal{B}(\mathbb{R})$ -measurable where $\mathcal{B}(\mathbb{R}) := \bigcap \left\{ \mathcal{G} \subset \mathcal{P}(\mathbb{R}): \sigma\text{-alg.} \right.$
 $\left. \wedge \forall a, b: [a, b] \in \mathcal{G} \right\}$
Borel σ -alg.

Lemma Let X be $\mathbb{R}$ -valued RV. Then $\mu_X(B) := P(X \in B)$, $B \in \mathcal{B}(\mathbb{R})$
 is a prob. measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

$$\text{Pf } \mu_X(\emptyset) = P(\emptyset) = 0, \quad \mu_X\left(\bigcup_{n \in \mathbb{N}} B_n\right) = P\left(X \in \bigcup_{n \in \mathbb{N}} B_n\right) = P\left(\bigcup_{n \in \mathbb{N}} \{X \in B_n\}\right) \\ = \dots = \sum_{n \in \mathbb{N}} \mu_X(B_n).$$

Def We call μ_X the distribution of X .

Ex X uniform on $[0,1]$... want $\mu_X([a,b]) = b-a \quad \forall 0 \leq a < b \leq 1.$

Note This problem is equivalent to ∞ -many coin tosses

$X \in [0,1)$. Define $Z_n := \begin{cases} 0 & \lfloor 2^n X \rfloor \text{ even} \\ 1 & \lfloor 2^n X \rfloor \text{ odd} \end{cases}$

$$X = \sum_{n=1}^{\infty} 2^{-n} Z_n$$

$$P\left(\bigcap_{i=1}^n \{Z_i = a_i\}\right) = P\left(\sum_{i=1}^n a_i 2^{-i} \leq X < \sum_{i=1}^n a_i 2^{-i} + 2^{-n}\right) = \frac{1}{2^n}$$