

Independence & product measures

Definition

We say that events A and B are **independent** if $P(A \cap B) = P(A)P(B)$.

Motivation: Conditional probability $P(A|B) := \frac{P(A \cap B)}{P(B)}$ gives $P(A|B) = P(A)$

Lemma

If A and B are independent then so are the events A and B^c (and thus also A^c and B^c).

Proof: Assume A and B to be independent. Then

$$\begin{aligned} P(A^c \cap B) &= P(B) - P(A \cap B) \\ &= P(B) - P(A)P(B) = [1 - P(A)]P(B) = P(A^c)P(B) \quad \square \end{aligned}$$

For more than two events the above gives only **pairwise independence**. For full independence more conditions are needed:

Definition

We say that events $\{A_\alpha : \alpha \in I\}$ are **independent** if

$$\forall J \subseteq I: J \text{ finite} \Rightarrow P\left(\bigcap_{\alpha \in J} A_\alpha\right) = \prod_{\alpha \in J} P(A_\alpha)$$

Example: Three events A_1, A_2 and A_3 are independent if

$$P(A_1 \cap A_2) = P(A_1)P(A_2)$$

$$P(A_1 \cap A_3) = P(A_1)P(A_3)$$

$$P(A_2 \cap A_3) = P(A_2)P(A_3)$$

and

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$

Example: Let $\Omega := \{0, 1\}^n$ and for $\omega = (\omega_1, \dots, \omega_n) \in \Omega$ set $X_i(\omega) := \omega_i$. Take P to be uniform on Ω . Then

$$P(X_1 = a_1, \dots, X_n = a_n) = \frac{1}{2^n}$$

for all $(a_1, \dots, a_n) \in \Omega$. Hence the events $A_1, \dots, A_n$ defined by

$$A_i := \{X_i = a_i\} \quad i = 1, \dots, n$$

are independent.

Counterexample: The events

$$B_1 := \{X_1 \neq X_2\}, B_2 := \{X_2 \neq X_3\}, B_3 := \{X_3 \neq X_1\}$$

are pairwise independent because $P(B_i) = 1/2$ for $i = 1, 2, 3$, and

$$P(B_1 \cap B_2) = 1/4 = P(B_1)P(B_2) \text{ etc}$$

but not independent because $B_1 \cap B_2 \cap B_3 = \emptyset$ so

$$P(B_1 \cap B_2 \cap B_3) = 0 \neq P(B_1)P(B_2)P(B_3)$$

Definition

Random variables $\{X_\alpha : \alpha \in I\}$, defined over the same probability space, with X_α taking values in $(\Sigma_\alpha, \mathcal{G}_\alpha)$, are **independent** if for all $\{B_\alpha\}_{\alpha \in I}$ such that $\forall \alpha \in I: B_\alpha \in \mathcal{G}_\alpha$, the events

$$\{\{X_\alpha \in B_\alpha\} : \alpha \in I\}$$

are independent.

Definition

σ -algebras $\{\mathcal{G}_\alpha : \alpha \in I\}$ are **independent** if for all $\{A_\alpha\}_{\alpha \in I}$ such that $\forall \alpha \in I: A_\alpha \in \mathcal{G}_\alpha$ the events $\{A_\alpha : \alpha \in I\}$ are independent.

Note: In all cases, suffices to check factorization for all finite subcollections.

Lemma

$$\{X_\alpha: \alpha \in I\} \text{ independent} \Leftrightarrow \{\sigma(X_\alpha): \alpha \in I\} \text{ independent}$$

Proof: $\sigma(X_\alpha) = \{\{X_\alpha \in B\}: B \in \mathcal{G}_\alpha\}$

□

Lemma

Let $\{X_\alpha: \alpha \in I\}$ be random variables on $(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}_\alpha: \alpha \in I\}$ sub- σ -algebras of $\mathcal{F}$ such that $\forall \alpha \in I: X_\alpha$ is $\mathcal{F}_\alpha$ -measurable. Then

$$\{\mathcal{F}_\alpha: \alpha \in I\} \text{ independent} \Rightarrow \{X_\alpha: \alpha \in I\} \text{ independent}$$

Proof: $\forall \alpha \in I: \sigma(X_\alpha) \subseteq \mathcal{F}_\alpha$

□

Example: Suppose that $\{Y_{n,k}: n \geq 1 \wedge k = 1, \dots, m_n\}$ be independent random variables and, for each $n \geq 1$, let $h_n: \mathbb{R}^{m_n} \rightarrow \mathbb{R}$ be a Borel function. Claim:

$$\{h_n(Y_{n,1}, \dots, Y_{n,m_n}): n \geq 1\} \text{ are independent}$$

Need that $\{\sigma(Y_{n,1}, \dots, Y_{n,m_n}): n \geq 1\}$ are independent. For that we show ...

Lemma

Let $(\Omega, \mathcal{F}, P)$ be a probability space and, for each $\alpha \in I$, let $\mathcal{A}_\alpha \subseteq \mathcal{F}$ be a π -system.

Then

$$\{\mathcal{A}_\alpha : \alpha \in I\} \text{ independent} \Rightarrow \{\sigma(\mathcal{A}_\alpha) : \alpha \in I\} \text{ independent}$$

Proof: Suffices to prove for I finite, $I = \{1, \dots, n\}$. As $\Omega \in \mathcal{A}_\alpha$, it suffices to check factorization for n -tuples of events. For $J := \{2, \dots, n\}$ set

$$\mathcal{A}' := \left\{ \bigcap_{i \in J} A_i : (\forall i \in J : A_i \in \mathcal{A}_i) \right\}$$

Now, given any $A \in \mathcal{A}'$, check that

$$\mathcal{L}_A := \{B \in \mathcal{F} : P(B \cap A) = P(B)P(A)\} \text{ is a } \lambda\text{-system}$$

As $\mathcal{A}_1 \subseteq \mathcal{L}_A$ by assumption, Dynkin's π/λ -theorem gives $\sigma(\mathcal{A}_1) \subseteq \mathcal{L}_A$. Hence, $\sigma(\mathcal{A}_1)$ is independent of $\mathcal{A}'$. Induction: $\{\sigma(\mathcal{A}_i) : i = 1, \dots, n\}$ independent. $\square$

Note: $\mathcal{L}_A$ closed under disjoint unions but not necessarily under all unions.

Corollary

Let $\{\mathcal{G}_{\alpha,\beta}: \alpha \in I \wedge \beta \in J_\alpha\}$ be independent. Then so are $\{\sigma(\bigcup_{\beta \in J_\alpha} \mathcal{G}_{\alpha,\beta}): \alpha \in I\}$.

Proof: Each

$$\mathcal{A}_\alpha := \left\{ \bigcap_{\beta \in J} A_{\alpha,\beta} : J \subseteq J_\alpha \text{ finite} \wedge \forall \alpha \in J: A_{\alpha,\beta} \in \mathcal{G}_{\alpha,\beta} \right\}$$

is a π -system with $\sigma(\mathcal{A}_\alpha) \supseteq \sigma(\bigcup_{\beta \in J_\alpha} \mathcal{G}_{\alpha,\beta})$. □

Back to example: Since each $X_n := h_n(Y_{n,1}, \dots, Y_{n,m_n})$ is $\sigma(Y_{n,1}, \dots, Y_{n,m_n})$ measurable. The independence of $\{Y_{n,k}: n \geq 1 \wedge k = 1, \dots, m_n\}$ implies that $\{X_n: n \geq 1\}$ are independent. □

Corollary

Let $X_1, \dots, X_n$ be random variables such that, for all $b_1, \dots, b_n \in \mathbb{R}$,

$$P(X_1 \leq b_1, \dots, X_n \leq b_n) = \prod_{i=1}^n P(X_i \leq b_i)$$

Then $X_1, \dots, X_n$ are independent.

Proof: Each $\mathcal{A}_i := \{\{X_i \leq b_i\} : b_i \in \overline{\mathbb{R}}\}$ is a π -system with $\mathcal{A}_1, \dots, \mathcal{A}_n$ independent (using monotone limits for $b_i = \pm\infty$). Since $\sigma(\mathcal{A}_i) = \sigma(X_i)$, also $X_1, \dots, X_n$ are independent. $\square$

Note: Suffices to consider $b \in \mathbb{Q}$.

By Corollary, $\mathbb{R}$ -valued r.v.'s are independent iff their joint CDF is the product of individual CDFs. For their distribution we get:

Lemma

Given $\mathbb{R}$ -valued random variables $X_1, \dots, X_n$, write $\mu_{X_i}(A) := P(X_i \in A)$ for the distribution of X_i and $\mu_X(B) := P((X_1, \dots, X_n) \in B)$ for the distribution of the random vector $X := (X_1, \dots, X_n)$. Then $X_1, \dots, X_n$ are independent if and only if

$$\forall A_1, \dots, A_n \in \mathcal{B}(\mathbb{R}): \mu_X(A_1 \times \dots \times A_n) = \prod_{i=1}^n \mu_{X_i}(A_i)$$

Note: We want to call μ_X with such a property a **product measure**

Q: For what μ_{X_i} does such a measure exist? And is it unique?

Q: What σ -algebra is μ_X defined on?

Definition

Let $\{(\Omega_i, \mathcal{F}_i)\}_{i=1}^n$ be measurable spaces. Their **product space** is the measurable space $(\Omega, \mathcal{F})$ such that $\Omega := \times_{i=1}^n \Omega_i$ and **product σ -algebra**

$$\mathcal{F} := \sigma\left(\left\{\times_{i=1}^n A_i : (\forall i = 1, \dots, n: A_i \in \mathcal{F}_i)\right\}\right)$$

Notation: $\mathcal{F} := \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n$ or $\mathcal{F} := \bigotimes_{i=1}^n \mathcal{F}_i$

Note: $\mathcal{F}$ is the smallest σ -algebra generated by “rectangles”

Note: If X_i is a random variable on $(\Omega_i, \mathcal{F}_i)$, then we can realize it in the product space as

$$\tilde{X}_i(\omega) := X_i(\omega_i)$$

for $\omega = (\omega_1, \dots, \omega_n)$.

Theorem

Let $\{(\Omega_i, \mathcal{F}_i, \mu_i)\}_{i=1}^n$ be measure spaces (with μ_i possibly infinite) and set

$$\Omega := \bigtimes_{i=1}^n \Omega_i \quad \text{and} \quad \mathcal{F} := \bigotimes_{i=1}^n \mathcal{F}_i$$

Then there exists a measure μ on $(\Omega, \mathcal{F})$ such that

$$\mu\left(\bigtimes_{i=1}^n A_i\right) = \prod_{i=1}^n \mu_i(A_i)$$

holds for all $A_1, \dots, A_n$ with $\forall i = 1, \dots, n: A_i \in \mathcal{F}_i$. Moreover, if $\mu_1, \dots, \mu_n$ are all σ -finite, then μ is unique with this property.

Definition

In the uniqueness regime, we call such μ the **product measure** with notation $\mu := \mu_1 \otimes \dots \otimes \mu_n$ or $\mu := \bigotimes_{i=1}^n \mu_i$.

The set

$$\mathcal{S} := \left\{ \bigtimes_{i=1}^n A_i : (\forall i = 1, \dots, n : A_i \in \mathcal{F}_i) \right\}$$

is a semi-algebra. Define $\tilde{\mu}(\bigtimes_{i=1}^n A_i) := \prod_{i=1}^n \mu_i(A_i)$. We claim that $\tilde{\mu}$ is countably additive on $\mathcal{S}$. Indeed, for each $i = 1, \dots, n$ let $A_i \in \mathcal{F}_i$ and $\{B_{i,j}\}_{j \in \mathbb{N}} \in \mathcal{F}_i^{\mathbb{N}}$ be such that

$$\bigtimes_{i=1}^n A_i = \bigcup_{j \in \mathbb{N}} \bigtimes_{i=1}^n B_{i,j}$$

with the union disjoint. This means that, for all $(x_1, \dots, x_n) \in \Omega$,

$$1_{A_1}(x_1) \dots 1_{A_n}(x_n) = \sum_{j \in \mathbb{N}} 1_{B_{1,j}}(x_1) \dots 1_{B_{n,j}}(x_n)$$

Integrating over x_1 with respect to μ_1 and using the Monotone Convergence Theorem to swap the sum with the integral yields

$$\mu_1(A_1) 1_{A_2}(x_2) \dots 1_{A_n}(x_n) = \sum_{j \in \mathbb{N}} \mu_1(B_{1,j}) 1_{B_{2,j}}(x_2) \dots 1_{B_{n,j}}(x_n)$$

Proceeding similarly, we thus get

$$\mu_1(A_1) \dots \mu_n(A_n) = \sum_{j \in \mathbb{N}} \mu_1(B_{1,j}) \dots \mu_n(B_{n,j})$$

thus showing that $\tilde{\mu}$ is countably additive on $\mathcal{S}$. By the Hahn-Kolmogorov theorem, $\tilde{\mu}$ extends to a measure on $\sigma(\mathcal{S}) = \bigotimes_{i=1}^n \mathcal{F}_i$.

For uniqueness, observe that if for each $i = 1, \dots, n$ there is $\{\Omega_{i,k}\}_{k \geq 1} \in \mathcal{F}_i^{\mathbb{N}}$ with $\Omega_{i,k} \uparrow \Omega_i$ and $\mu_i(\Omega_{i,k}) < \infty$ for all $k \geq 1$, then for

$$\tilde{\Omega}_k := \bigtimes_{i=1}^n \Omega_{i,k}$$

we have $\tilde{\Omega}_k \uparrow \Omega$ and $\tilde{\mu}(\tilde{\Omega}_k) < \infty$ for all $k \geq 1$. Hence $\tilde{\mu}$ is σ -finite on $\mathcal{S}$ and so the extension is unique by Dynkin's π/λ -theorem. $\square$

Take away message:

Any finite number of random variables can be realized as independent in a suitable product space.

Next time:

- Consequences: Fubini-Tonelli theorem
- Arbitrarily indexed product spaces
- Infinite families of random variables
- Infinite product measures