

Thm (DCT) Let $\{f_n\}_{n \in \mathbb{N}}$, f be meas. functions $\Omega \rightarrow \overline{\mathbb{R}}$ st.
 $\exists g: \Omega \rightarrow [0, \infty]$ meas. with:

- 1) $f_n \rightarrow f$ pointwise
- 2) $\forall n \in \mathbb{N}: |f_n| \leq g$
- 3) $\int g d\mu < \infty$.

Then $\int f_n d\mu \rightarrow \int f d\mu$.

(Note: all integrals exist.)

Pf $g + f_n \geq 0$ so Fatou's lemma:

$$\begin{aligned} \int g d\mu + \int f d\mu &= \int \liminf_{n \rightarrow \infty} (g + f_n) d\mu \\ &\leq \liminf_{n \rightarrow \infty} \int (g + f_n) d\mu = \int g d\mu + \liminf_{n \rightarrow \infty} \int f_n d\mu \end{aligned}$$

$g - f_n \geq 0$ so again:

$$\begin{aligned} \int g d\mu - \int f d\mu &= \int \liminf_{n \rightarrow \infty} (g - f_n) d\mu \\ &\leq \liminf_{n \rightarrow \infty} \int (g - f_n) d\mu = \int g d\mu - \limsup_{n \rightarrow \infty} \int f_n d\mu \end{aligned}$$

Combining:

$$\int f d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu \leq \limsup_{n \rightarrow \infty} \int f_n d\mu \leq \int f d\mu$$

So $\int f_n d\mu \rightarrow \int f d\mu$. $\square$

We used:

Lemma If $\int |f| d\mu < \infty$, then $\int f d\mu$ exists and $|\int f d\mu| \leq \int |f| d\mu$.

Pf: $f = f^+ - f^-$, $|f| = f^+ + f^-$ so $f^\pm \leq |f|$ so $\int |f| d\mu < \infty \Rightarrow \int f^\pm d\mu$ defined.

Now $-|f| \leq f \leq |f|$ gives $-\int |f| d\mu \leq \int f d\mu \leq \int |f| d\mu$. $\square$

Def: A measurable $f: \Omega \rightarrow \overline{\mathbb{R}}$ is absolutely integrable (w.r.t. μ) if $\int |f| d\mu < \infty$.

Thm (Bounded Convergence Thm) Let $\{f_n\}_{n \in \mathbb{N}}$, f be meas. functions $\Omega \rightarrow \mathbb{R}$ s.t.

- 1) $f_n \rightarrow f$ ptwise
- 2) $\exists M \geq 0 \forall n \in \mathbb{N}: |f_n| \leq M$
- 3) $\mu(\Omega) < \infty$.

Then $\int f_n d\mu \rightarrow \int f d\mu$.

Pf: $g := M$ then $\int g d\mu = M \mu(\Omega) < \infty$.
Now use DCT. $\square$

Def $f: \Omega \rightarrow \mathbb{R}$ meas. is square integrable if $\int f^2 d\mu < \infty$.

Some inequalities:

Lemma (Cauchy-Schwarz) Let $f, g: \Omega \rightarrow \mathbb{R}$ be meas. s.t. $\int f^2 d\mu < \infty \wedge \int g^2 d\mu < \infty$.

Then $\int fg d\mu$ exists and $|\int fg d\mu| \leq [\int f^2 d\mu]^{1/2} [\int g^2 d\mu]^{1/2}$

Pf: If we show the i.e.g. for $f, g \geq 0$, we get fg is abs. integrable and so $\int fg d\mu$ exists by prev. lemma.
The i.e.g. ab $\leq \frac{a}{2} + \frac{b}{2}$ shows that $\int fg d\mu \leq \frac{1}{2} \int f^2 d\mu + \frac{1}{2} \int g^2 d\mu < \infty$.
Now use $\forall \lambda \in \mathbb{R}$:

$0 \leq \int (f - \lambda g)^2 d\mu = \int f^2 d\mu - 2\lambda \int fg d\mu + \lambda^2 \int g^2 d\mu$. Can only be true if $D := (2 \int fg d\mu)^2 - 4(\int f^2 d\mu)(\int g^2 d\mu) \leq 0$
 $\Rightarrow$ inequality

Lemma (Hölder inequality) Let $p, q \in [1, \infty)$ be s.t. $\frac{1}{p} + \frac{1}{q} = 1$.
 Let $f, h: \Omega \rightarrow \overline{\mathbb{R}}$ be meas. s.t. $\int |f|^p d\mu < \infty \wedge \int |h|^q d\mu < \infty$.
 Then $\int f h d\mu$ exists and $|\int f h d\mu| \leq [\int |f|^p d\mu]^{1/p} [\int |h|^q d\mu]^{1/q}$.

PP As before, may assume $f, h \geq 0$, $0 < \int |f|^p d\mu < \infty \wedge 0 < \int |h|^q d\mu < \infty$.

AMGM: $\forall a, b \geq 0: ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

Set $\tilde{f}(x) := \frac{f(x)}{[\int |f|^p d\mu]^{1/p}}$, $\tilde{h}(x) := \frac{h(x)}{[\int |h|^q d\mu]^{1/q}}$. Then $\int \tilde{f} \tilde{h} d\mu \leq \int \frac{\tilde{f}^p}{p} d\mu + \int \frac{\tilde{h}^q}{q} d\mu = \frac{1}{p} + \frac{1}{q} = 1$.

Now LHS = $\int f h d\mu / [\int f^p d\mu]^{1/p} [\int h^q d\mu]^{1/q}$ $\square$

Def A meas. $f: \Omega \rightarrow \overline{\mathbb{R}}$ is p-integrable if $\int |f|^p d\mu < \infty$.

Back to probability

Def Let X be a RV on $(\Omega, \mathcal{F}, P)$. Its expectation EX is

$$EX := \int X dP = \int_{\Omega} X(\omega) P(d\omega)$$

provided integral exists.

Thm (Change of variables formula) Let X be RV on $(\Omega, \mathcal{F}, P)$ taking val's in $(\Sigma, \mathcal{G})$.
 Then for any $\mathcal{G}/\mathcal{B}(\mathbb{R})$ -meas. function $h: \Sigma \rightarrow \mathbb{R}$:

$$E(h(X)) = \int h \, d\mu_X = \int h(x) \mu_X(dx)$$

where μ_X is the distribution of X on $(\Sigma, \mathcal{G})$, provided at least one of integrals exists.

Pf by monotone-class argument. (HW4)

Lemma (Jensen's inequality) Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ ($\text{Dom}(\varphi) = \mathbb{R}$) be convex.

Then for any RV X s.t. $E|X| < \infty$ we have $E\varphi(X)$ exists and $E\varphi(X) \geq \varphi(EX)$.

Pf: $\exists c_1, c_2 > 0 \forall x \in \mathbb{R}: \varphi(x) \geq -c_1 - c_2|x|$ so $E\varphi(X) < \infty$ if $E|X| < \infty$.

claim: Set $\mathcal{L} := \{(a, b) \in \mathbb{R}^2: (\forall x \in \mathbb{R}: \varphi(x) \geq ax + b)\}$. Then $\forall x \in \mathbb{R}: \varphi(x) = \sup_{(a, b) \in \mathcal{L}} (ax + b)$.

Pf: $\varphi(x) \geq \varphi(x_0) + \frac{d\varphi}{dx}(x_0)(x - x_0)$ wtl = when $x = x_0$. Take $a = \frac{d\varphi}{dx}(x_0)$, $b := \varphi(x_0) - x_0 \frac{d\varphi}{dx}(x_0) \Rightarrow (a, b) \in \mathcal{L}$

$$\text{Now } E\varphi(X) \geq \sup_{(a, b) \in \mathcal{L}} E(aX + b) = \sup_{(a, b) \in \mathcal{L}} (aEX + b) = \varphi(EX). \quad \square$$