

MATH275A: HW#3

Deferred to Mon, Oct 26

The numbering refers to online version of the textbook

Problem 1: Recall the concept of an outer measure μ^* and the notation $\mathcal{F}(\mu^*)$ for the class of Carathéodory measurable sets. An outer measure μ^* on Ω is said to be *regular* if for each $A \subseteq \Omega$ there is $B \in \mathcal{F}(\mu^*)$ such that $A \subseteq B$ and $\mu^*(B) = \mu^*(A)$. Prove that if μ^* is regular, then

$$\forall A \subseteq \Omega: A \in \mathcal{F}(\mu^*) \Leftrightarrow \mu^*(\Omega) = \mu^*(A) + \mu^*(A^c)$$

Note: For finite μ^* , this relates the concept of measurability to equality of the outer and inner measure, with the latter defined as $\mu_*(A) := \mu^*(\Omega) - \mu^*(A)$. The outer measures constructed in class were all regular.

Problem 2: Give an example of:

- (1) a σ -finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which is not a Stieltjes measure (i.e., for which there is no $F: \mathbb{R} \rightarrow \mathbb{R}$ such that $\mu((a, b]) = F(b) - F(a)$ for all $a < b$).
- (2) two measures μ_1 and μ_2 on a measurable space $(\Omega, \mathcal{F})$ such that $\mu_1 = \mu_2$ on an algebra $\mathcal{A} \subseteq \mathcal{F}$ yet $\mu_1 \neq \mu_2$ on $\sigma(\mathcal{A})$.
- (3) a non-decreasing bounded $F: \mathbb{R} \rightarrow \mathbb{R}$ along with an interval $(a, b]$ and disjoint intervals $\{(a_i, b_i]\}_{i \in \mathbb{N}}$ such that

$$(a, b] = \bigcup_{i \in \mathbb{N}} (a_i, b_i] \quad \text{yet} \quad F(b) - F(a) > \sum_{i \in \mathbb{N}} [F(b_i) - F(a_i)]$$

Problem 3: A d -dimensional random vector on $(\Omega, \mathcal{F})$ is an (everywhere defined) $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$ -measurable map $X \mapsto \mathbb{R}^d$. Do the following:

- (1) Writing $X = (X_1, \dots, X_n)$, prove that each X_i is an $\mathbb{R}$ -valued random variable.
- (2) Then show that, conversely, for any $\mathbb{R}$ -valued random variables $X_1, \dots, X_n$ on $(\Omega, \mathcal{F})$, the object $X := (X_1, \dots, X_n)$ is a random vector.

Problem 4: EX 1.4.1, PAGE 24

Problem 5: EX 1.4.3, PAGE 24

Problem 6: EX 1.5.2, PAGE 28

Problem 7: EX 1.5.8, PAGE 28

Problem 8: EX 1.5.10, PAGE 28
