

MATH275A: HW#2

Due: Mon, Oct 12

The numbering refers to online version of the textbook

Problem 1: Given sets $\{A_n\}_{n \geq 1}$ and A , we write $A_n \rightarrow A$ if both $\limsup_{n \rightarrow \infty} A_n := \bigcap_{n \geq 1} \bigcup_{k > n} A_k$ and $\liminf_{n \rightarrow \infty} A_n := \bigcup_{n \geq 1} \bigcap_{k > n} A_k$ equal to A . Prove that

$$A_n \rightarrow A \Leftrightarrow 1_{A_n} \rightarrow 1_A \text{ pointwise}$$

Then show that also $A_n \rightarrow A$ implies $\mu(A_n) \rightarrow \mu(A)$ for all finite measures μ .

Problem 2: EX 9, PAGE 1.1.3

Problem 3: Given a non-empty set Ω , let $\mathcal{S} \subseteq \mathcal{P}(\Omega)$ be a semialgebra — i.e., a class of sets that is closed under finite intersections, contains $\emptyset$ and Ω and with $A \in \mathcal{S}$ implying that A^c is a finite disjoint union of sets from $\mathcal{S}$. Prove that

$$\mathcal{A} := \left\{ \bigcup_{i=1}^n A_i : n \geq 1 \wedge \{A_i\}_{i=1}^n \in \mathcal{S}^n \wedge \{A_i\}_{i=1}^n \text{ disjoint} \right\}$$

is an algebra on Ω .

Problem 4: Continuing in the setup of the previous problem, next let us consider a function $\mu_0: \mathcal{S} \rightarrow [0, \infty]$ that is

(1) finitely additive on $\mathcal{S}$:

$$\forall n \geq 1 \forall A_1, \dots, A_n \in \mathcal{S}: \{A_i\}_{i=1}^n \text{ disjoint} \wedge \bigcup_{i=1}^n A_i \in \mathcal{S} \Rightarrow \mu_0\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mu_0(A_i)$$

(2) countably subadditive on $\mathcal{S}$:

$$\forall \{A_i\}_{i \in \mathbb{N}} \in \mathcal{S}^{\mathbb{N}}: \bigcup_{i \geq 1} A_i \in \mathcal{S} \Rightarrow \mu_0\left(\bigcup_{i \in \mathbb{N}} A_i\right) \leq \sum_{i \in \mathbb{N}} \mu_0(A_i)$$

Then prove that, for all disjoint $\{A_i\}_{i=1}^n$ and all disjoint $\{B_j\}_{j=1}^m$,

$$\bigcup_{i=1}^n A_i = \bigcup_{j=1}^m B_j \Rightarrow \sum_{i=1}^n \mu_0(A_i) = \sum_{j=1}^m \mu_0(B_j)$$

The common value then defines $\bar{\mu}_0: \mathcal{A} \rightarrow [0, \infty]$ by $\bar{\mu}_0(\bigcup_{i=1}^n A_i) := \sum_{i=1}^n \mu_0(A_i)$. Now show that $\bar{\mu}_0$ is finitely additive and countably subadditive on $\mathcal{A}$ and satisfies

$$\forall A \in \mathcal{S}: \bar{\mu}_0(A) = \mu_0(A)$$

Note: This all means that μ_0 extends to a premeasure on the algebra $\mathcal{A}$ above. The Hahn-Kolmogorov Theorem then extends $\bar{\mu}_0$ further to a measure on $\sigma(\mathcal{S})$.

Problem 5: EX 1.2.4, PAGE 14

Problem 6: EX 1.3.7, PAGE 17

Problem 7: A class of sets $\mathcal{M} \subseteq \mathcal{P}(\Omega)$ is said to be a monotone class if

(1) $\forall \{A_n\} \in \mathcal{M}^{\mathbb{N}} \forall A \in \mathcal{P}(\Omega): A_n \uparrow A \text{ implies } A \in \mathcal{M}$

(2) $\forall \{A_n\} \in \mathcal{M}^{\mathbb{N}} \forall A \in \mathcal{P}(\Omega): A_n \downarrow A \text{ implies } A \in \mathcal{M}$

Prove that if $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ an algebra, then

$$\mathcal{A} \subseteq \mathcal{M} \quad \Rightarrow \quad \sigma(\mathcal{A}) \subseteq \mathcal{M}$$

(This is the Monotone Class Theorem for Sets which is similar to Dynkin's π/λ -Theorem except that here we ask somewhat less from $\mathcal{M}$ and somewhat more from $\mathcal{A}$.)

Problem 8: EX A.1.1, PAGE 458
