Sobolev inequalities

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Outline:

- Gagliardo-Nierenberg-Sobolev inequality
- $L^1$-Sobolev and $L^2$-Sobolev inequalities
- Sobolev embedding theorems
- Compact embedding and Rellich-Kondrachov’s theorem
- Boost in regularity via Moser iteration
Morrey’s inequality: regularity for $f \in W^{1,p}(\mathbb{R}^d)$ with $p > d$.

Q: What happens for $p < d$?

**Theorem**

For all $d \geq 2$ and all $p \in [1, d)$ there is $c(d, p) \in (0, \infty)$ such that

$$\forall f \in C_c^\infty(\mathbb{R}^d) : \quad \|f\|_{p^*} \leq c(d, p)\|\nabla f\|_p$$

where $p^*$ is the Sobolev conjugate of $p$ defined by

$$p^* := \frac{pd}{d - p}$$

Proved independently by E. Gagliardo and L. Nirenberg for $p = 1$ and by S.L. Sobolev for $1 < p < d$. 

Gagliardo-Nirenberg-Sobolev inequality
First some remarks

\[ p^* := \frac{pd}{d-p} \] the only index for which this can hold. Indeed, take \( f_t(x) := f(x/t) \). Then

\[ \|f_t\|_{p^*} = t^{\frac{d}{p^*}} \|f\|_{p^*} \land \|\nabla f_t\|_p = t^{\frac{d}{p}-1} \|\nabla f\|_p \]

If \( \|f\|_{p^*} \leq c \|\nabla f\|_p \) is to hold for \( f_t \) for all \( t > 0 \), then we must have

\[ \frac{1}{p^*} = \frac{1}{p} - \frac{1}{d} \]

This forces \( p \leq d \). The case \( p = d \) excluded in \( d \geq 2 \) by

\[ f_\epsilon(x) := \log(\epsilon + |x|) g(x) \]

with \( g \in C^\infty_c(\mathbb{R}^d) \) s.t. \( \text{supp}(g) \subseteq B(0, 1/2) \) and \( g(0) \neq 0 \). Then

\[ \|\nabla f_\epsilon\|_d \leq c (\log(1/\epsilon))^{1/d} \] yet \( \|f_\epsilon\|_\infty \geq |g(0)| \log(1/\epsilon) \)

so \( f \mapsto \|f\|_\infty / \|\nabla f\|_d \) not bounded on \( C^\infty_c(\mathbb{R}^d) \).
For $d = 1$ the inequality holds by
\[ \forall f \in C_c^\infty(\mathbb{R}): \quad \|f\|_\infty \leq \|f'\|_1 \]

Proved by writing $f(x) = \int_{(\infty,x]} f'd\lambda$ (as $\text{supp}(f)$ compact).

The case $p = 1$ fundamental for all $d \geq 1$:

**Lemma**

Let $d \geq 2$ and suppose there is $c(d) \in (0, \infty)$ such that
\[ \forall f \in C_c^\infty(\mathbb{R}^d): \quad \|f\|_{d \over d-1} \leq c(d)\|\nabla f\|_1 \]

Then Sobolev inequality holds for all $p \in [1, d)$ (and $p^*$ as in (1)) with
\[ c(d, p) := c(d)\frac{d-1}{d-p}p \]
First extend $L^1$-Sobolev inequality to $f \in C^1_c(\mathbb{R}^d)$. Pick $r \geq 1$ and, given $f \in C^\infty_c(\mathbb{R}^d)$, set $\tilde{f} := f |f|^{r-1}$. Then $\nabla \tilde{f} = rf |f|^{r-2} \nabla f$ (and so $\tilde{f} \in C^1_c(\mathbb{R}^d)$) which shows

$$
\|f\|^{r\frac{d}{d-1}} = \|\tilde{f}\|^{\frac{d}{d-1}} \leq c \|\nabla \tilde{f}\|_1 = cr \|f |f|^{r-2} \nabla f\|_1 = cr \langle |f|^{r-1}, |\nabla f| \rangle
$$

Hölder’s inequality with indices $\tilde{p}, \tilde{q} \in [1, \infty]$ gives

$$
\langle |f|^{r-1}, |\nabla f| \rangle \leq \|f|^{r-1}\|_{\tilde{p}} \|\nabla f\|_{\tilde{q}} = \|f\|^{r-1}_{\tilde{p}(r-1)} \|\nabla f\|_{\tilde{q}}
$$

Now set $r := \frac{d-1}{d-p} p$ and observe that this implies

$$
\tilde{p}(r-1) = r \frac{d}{d-1} = p^* \quad \land \quad \tilde{q} = p
$$

Putting these together we get the claim. \qed
Proof of $L^1$-Sobolev inequality

We thus focus on the proof of $L^1$-Sobolev inequality. This will be derived from

$$\forall f \in C_c^\infty(\mathbb{R}^d): \quad \|f\|_1 \leq \prod_{i=1}^{d} \|\partial_i f\|_1^{1/d}$$

from which we get $L^1$-Sobolev via

$$\prod_{i=1}^{d} \|\partial_i f\|_1^{1/d} \leq \frac{1}{d} \left( \sum_{i=1}^{d} |\partial_i f|_1 \right) \leq \frac{1}{\sqrt{d}} \left( \sum_{i=1}^{d} |\partial_i f|^2 \right)^{1/2} = \frac{1}{\sqrt{d}} \|\nabla f\|_1$$

based on arithmetic-geometric/Cauchy-Schwarz inequalities.

The above inequality holds for $d = 1$ directly. We prove the other cases by induction. Assume it holds for dimension $d - 1$ and let us prove it for $d$ ...
Proof of $L^1$-Sobolev inequality continued ...

Write points of $\mathbb{R}^d$ as $(x, z)$ where $x \in \mathbb{R}$ and $z \in \mathbb{R}^{d-1}$. Set $\theta := \frac{d-2}{d-1}$. Then $\frac{d}{d-1} = \theta \frac{d-1}{d-2} + (1 - \theta)$ and so by Hölder:

$$\forall x \in \mathbb{R}^d : \quad \int |f(x, z)| \frac{d}{d-1} dz \leq \left( \int |f(x, z)| \frac{d-1}{d-2} dz \right)^{\frac{d-2}{d}} \left( \int |f(x, z)| dz \right)^{\frac{1}{d-1}}$$

Now

$$\forall x \in \mathbb{R}^d : \quad \int |f(x, z)| dz \leq \| \partial_1 f \|_1$$

Induction assumption and multivariate Hölder in turn yield

$$\int \left( \int |f(x, z)| \frac{d-1}{d-2} dz \right)^{\frac{d-2}{d-1}} dx \leq \int \prod_{i=2}^d \| \partial_i f(x, \cdot) \|_1 \frac{1}{d-1} dx \leq \prod_{i=2}^d \| \partial_i f \|_1 \frac{1}{d-1}$$

Putting together we get

$$\int |f| \frac{d}{d-1} d\lambda \leq \prod_{i=1}^d \| \partial_i f \|_1 \frac{1}{d-1}$$

which readily gives the claim. \qed
Corollary

Let $d \geq 2$. Then

$$\forall p \in [1, d): \quad W^{1,p}(\mathbb{R}^d) \subseteq \bigcap_{p \leq p' \leq \frac{dp}{d-p}} L^{p'}(\mathbb{R}^d).$$

Proof: Sobolev inequality gives $W^{1,p}(\mathbb{R}^d) \subseteq L^{p^*}$. By definition $W^{1,p}(\mathbb{R}^d) \subseteq L^p$. Now apply interpolation of $L^p$ norms. \qed
The case $p = 2$ frequently used, but excluded in $d = 2$. This is by including $L^p$-norm on r.h.s. as in:

**Theorem**

Let $d \geq 2$. For each $q \in [2, \frac{2d}{d-2})$ (where $\frac{2d}{d-2}$ is interpreted as $\infty$ when $d = 2$) there is $\tilde{c}(d, q) \in (0, \infty)$ such that

$$\forall f \in C_\infty^\infty (\mathbb{R}^d) : \|f\|_q \leq \tilde{c}(d, q) (\|f\|_2 + \|\nabla f\|_2)$$

$\tilde{c}(d, q)$ is bounded in $d \geq 3$, the bound extends to $q = \frac{2d}{d-2}$. 
$L^2$-Sobolev in $d \geq 3$

Let $q \in \left[2, \frac{2d}{d-2}\right) = p^*$ for $p = 2$. As $p^* > p$, find $\theta \in [0, 1]$ such that

$$\frac{1}{q} = \frac{\theta}{p^*} + \frac{1-\theta}{2}.$$  

Interpolation of $L^p$-norms:

$$\|f\|_q \leq \|f\|_{p^*}^{\theta} \|f\|_2^{1-\theta} \leq c(d, 2) \|\nabla f\|_2^{\theta} \|f\|_2^{1-\theta}$$

$$\leq c(d, 2) \left(\theta \|\nabla f\|_2 + (1 - \theta) \|f\|_2\right)$$

where arithmetic-geometric inequality used in the last step.  \Box
Need some Fourier transform facts:

**Lemma**

For each \( f \in C_\infty_c(\mathbb{R}^d) \),

\[
\int (1 + 4\pi^2|k|^2)|\hat{f}(k)|^2 \, dk = \|f\|_2^2 + \|\nabla f\|_2^2
\]

Moreover, for all \( f \in C_\infty_c(\mathbb{R}^d) \) there is \( c \in (0, \infty) \) s.t.

\[
\forall k \in \mathbb{R}^d : \quad |\hat{f}(k)| \leq \frac{c}{(1 + 4\pi^2|k|^2)^{d/2}}.
\]

In particular, \( \hat{f} \in L^1 \).

Proof: Write \( \hat{\nabla f}(k) = -2\pi ik\hat{f}(k) \) and use that Fourier transform is an isometry in \( L^2 \). For second part, take \( \tilde{f} := (1 - \Delta)^d f \) to get above with \( c := \|\hat{f}\|_{\infty} \leq \|\hat{f}\|_1 \).
Proof of $L^2$-Sobolev in $d \geq 2$

Take $q \in (2, \frac{2d}{d-2})$ and let $p \in [1, 2)$ be the Hölder dual and let $f \in C_c^\infty(\mathbb{R}^d)$. Then $x \mapsto f(-x)$ is thus the Fourier transform of $\hat{f}$. Hausdorff-Young inequality gives

$$\|f\|_q \leq \|\hat{f}\|_p = \left( \int (1 + 4\pi |k|^2)^{p/2} |\hat{f}(k)|^p (1 + 4\pi |k|^2)^{-p/2} \, dk \right)^{1/p}$$

$$\leq \left( \int (1 + 4\pi^2 |k|^2) |\hat{f}(k)|^2 \, dk \right)^{1/2} \left( \int (1 + 4\pi^2 |k|^2)^{-\frac{p}{2-p}} \, dk \right)^{\frac{1}{p} - \frac{1}{2}}.$$

by Hölder’s inequality with parameters $2/p$ and $\frac{2}{2-p}$. The integral converges when $\frac{2p}{2-p} > d$ which is equivalent to $q < \frac{2d}{d-2}$. The claim follows from above lemma. \qed
Corollary

For all $d \geq 2$,

\[ W^{1,2}(\mathbb{R}^d) \subseteq \bigcap_{2 \leq q < \frac{2d}{d-2}} L^q(\mathbb{R}^d). \]
Theorem (Rellich-Kondrachov)

Let $d \geq 2$ and let $O \subseteq \mathbb{R}^d$ be bounded and open. Given $p \in [1, d)$ let $q \in [1, p^*)$, where $p^* := \frac{dp}{d-p}$. Then every non-empty bounded set $C \subseteq W_0^{1,p}(O)$ is has a compact closure in $L^q$. 
Criterion of compactness in $L^p$

Lemma (Kolmogorov-Riesz compactness theorem)

Let $p \in [1, \infty)$ and let $C \subseteq L^p(\mathbb{R}^d)$ be non-empty. Then $C$ is totally bounded in $L^p(\mathbb{R}^d)$ if and only if it is bounded,

$$\sup_{f \in C} \|f\|_p < \infty$$

and tight

$$\lim_{r \to \infty} \sup_{f \in C} \|f1_{B(0,r)^c}\|_p = 0$$

and, denoting $f_h(x) := f(x + h)$ for $h \in \mathbb{R}^d$, obeys

$$\lim_{\epsilon \downarrow 0} \sup_{|h| < \epsilon} \| f_h - f \|_p = 0.$$
Proof of Lemma

Suffices to show that $C$ contains a finite $\delta$-net for each $\delta > 0$. Let $\chi_\varepsilon$ by our standard mollifier and set

$$C_\varepsilon := \{(f1_{B(0,1/\varepsilon)}) \ast \chi_\varepsilon : f \in C\}.$$ 

As $\|\chi_\varepsilon\|_1 = 1$, Minkowski’s integral inequality implies

$$\|(f1_{B(0,1/\varepsilon)}) \ast \chi_\varepsilon - f\|_p \leq \|(f1_{B(0,1/\varepsilon)}) - f\|_p + \sup_{|h| < \varepsilon} \|f_h - f\|_p$$

So suffices to find a $\delta$-net in $C_\varepsilon$. But Hölder gives

$$\|\nabla (f1_{B(0,1/\varepsilon)} \ast \chi_\varepsilon)\|_\infty \leq \|f\|_p \|\nabla \chi_\varepsilon\|_q$$

and so, since $\|(f1_{B(0,1/\varepsilon)}) \ast \chi_\varepsilon\|_\infty \leq \|f1_{B(0,1/\varepsilon)}\|_1 \leq c\|f\|_p$, we get

$C_\varepsilon$ is an equicontinuous family

As supp($f1_{B(0,1/\varepsilon)} \ast \chi_\varepsilon$) $\subseteq B(0, \varepsilon + 1/\varepsilon)$, the claim follows from Arzelà-Ascoli in sup-norm and then also $L^p$-norm by compact support. For converse, see the notes. □
Proof of Rellich-Kondrachov’s Theorem

Let \( C \subseteq W^{1,2}_0(O) \). Tightness trivial as \( O \) bounded. Sobolev implies

\[
\|f\|_q \leq \lambda(O)^{\frac{1}{q} - \frac{1}{p^*}} \|f\|_{p^*} \leq c\lambda(O)^{\frac{1}{q} - \frac{1}{p^*}} \|\nabla f\|_p
\]

so \( C \) bounded also in \( L^q \).

For the last condition find \( \theta \) s.t. \( \theta \in [0,1] \) by \( \frac{1}{q} = 1 - \theta + \frac{\theta}{p^*} \). Then

\[
\|f_h - f\|_q \leq \|f_h - f\|_1^{1-\theta} \|f_h - f\|_{p^*}^{\theta} \leq (2c)^\theta \|f_h - f\|_1^{1-\theta} \|\nabla f\|_p^\theta.
\]

As \( 1 - \theta > 0 \) because \( q < p^* \), it suffices to show

\[
\limsup_{h \to 0} \sup_{f \in C} \|f_h - f\|_1 = 0
\]

Writing

\[
|f_h(x) - f(x)| \leq |h| \int \left| \nabla f(x + th) \right| dt
\]

we get

\[
\|f_h - f\|_1 \leq |h| \text{diam}(O) \|\nabla f\|_1 \leq |h| \text{diam}(O) \lambda(O)^{1 - \frac{1}{p}} \|\nabla f\|_p
\]

Since \( \sup_{f \in C} \|\nabla f\|_p < \infty \), we are done. \( \square \)
Let $d \geq 2$, pick $O \subseteq \mathbb{R}^d$ bounded open and consider

$$\inf \left\{ \frac{1}{2} \| \nabla f \|_2^2 : f \in C_c^\infty (O) \land \|f\|_r = 1 \right\}$$

We will take $r > 1$ in what follows even though $r = 1$ very interesting too.

For $r = 2$, this describes base frequency of a drum shaped as $O$.

Q: Is there a minimizing $f$ (with $f \in L^r$ and $\nabla f \in L^2$)?

Q: And if so, how regular is it?
Lemma

For $r \in (1, \infty)$, there exists $f \in W^{1,2}_0(O)$ with $\|f\|_r = 1$ such that

$$\frac{1}{2} \|\nabla f\|_2^2 = \inf\left\{ \frac{1}{2} \|\nabla h\|_2^2 : h \in C_c^\infty(O) \land \|h\|_r = 1 \right\}$$

Moreover, there is $\beta \geq 0$ such that $f$ satisfies

$$\Delta f = -\beta f |f|^{r-2}$$

where the Laplacian is taken in the sense of weak derivatives; i.e.,

$$\forall \phi \in C_c^\infty(O) : \int (-\Delta \phi) f d\lambda = \beta \int \phi f |f|^{r-2} d\lambda.$$
Proof of Lemma

Let \( \{f_n\}_{n \geq 1} \subseteq C_\infty^0(O) \) be minimizing sequence. Then \( \sup_{n \geq 1} \|\nabla f_n\|_2 < \infty \) so Sobolev inequality gives

\[
\forall q \in \left[1, \frac{2d}{d-2}\right]: \quad \sup_{n \geq 1} \|f_n\|_q < \infty
\]

This includes \( q = 2 \) and so \( \{f_n\}_{n \geq 1} \) is bounded in \( W_0^{1,2}(O) \) and thus also in \( W_0^{1,p}(O) \) for all \( p \in [1, 2) \).

Rellich-Kondrachov theorem shows (for a subsequence if needed) that

\[
\forall q' \in \left[1, \frac{2d}{d-2}\right]: \quad f_n \rightharpoonup f \text{ in } L^{q'}
\]

(Limit same for all \( q' \).) This includes \( q' = 2 \) and so \( f_n \rightharpoonup f \) in \( L^2 \).

Integrate w.r.t. \( \phi \in C_\infty^0(O) \) to get

\[
\int \phi(\nabla f_n) \, d\lambda = \int (\nabla \phi) f_n \, d\lambda \rightarrow \int (\nabla \phi) f \, d\lambda
\]

As \( C_\infty^0(O) \) is dense in \( L^2 \) and \( \{\nabla f_n\}_{n \geq 1} \) bounded in \( L^2 \), \( f \) is weakly differentiable and \( \nabla f_n \rightharpoonup \nabla f \) weakly in \( L^2 \). Now …
...lower semicontinuity of $L^p$ norms under weak limits shows

$$\|\nabla f\|_2 \leq \liminf_{n \to \infty} \|\nabla f_n\|_2.$$  

So $\frac{1}{2}\|\nabla f\|_2^2 \leq \inf\{\ldots\}$. Can have “<” because $f$ can be approximated by $C_c^\infty(O)$-functions. So $f$ is a minimizer.

To show that $f$ solves PDE, take $\tilde{f} := (f + t\phi)/\|f + t\phi\|_1$ for $\phi \in C_c^\infty(O)$ to get

$$\|\nabla f + t\nabla \phi\|_2^2 \geq \|\nabla f\|_2^2 + \|\nabla f\|_2^2 (\|f + t\phi\|_r^2 - \|f\|_r^2)$$

Now expand in lowest order in $t$ to get

$$\int (\nabla f) \cdot (\nabla \phi) d\lambda = 2r\|\nabla f\|_2^2 \int f|f|^{-2}\phi \, d\lambda$$

Finally, set $\beta := 2r\|\nabla f\|_2^2$. \qed
Remarks

- \( \beta \) is closely related to the value of the infimum
- for \( r = 2 \) we get an *eigenvalue problem* for the Laplacian operator on \( W^{1,2}_0(O) \),

\[
-\Delta f = \beta f
\]

May have multiple solutions, even with same \( \beta \).
- \( r = 1 \) excluded because formal derivative of \( f \mapsto |f| \) does not capture everything; instead we get

\[
-\Delta f = \beta \text{sgn}(f) + \beta' \delta_{f(0)}
\]

Needs theory of distributions
Lemma

Let \( r \in (1, \frac{2d}{d-2}) \) and \( \beta \geq 0 \). Then there is \( c(d, r, \beta, O) \in (0, \infty) \) such that any \( f \in W^{1,2}_0(O) \) that solves

\[-\Delta f = \beta |f|^{r-2}\]

weakly obeys

\[\|f\|_\infty \leq c(d, r, \beta, O) \|f\|_2\]

In particular, \( f \in L^\infty \).
Proof of Moser’s iterative estimate

Pick $q \in (2, \frac{2d}{d-2})$ with $q > r$ and let $f \in W^{1,2}(O)$ solves the PDE. Assume $f \in L^{2s}$ for some $s \geq 1$. Then $f_s := f|f|^{s-1}$ obeys $f_s \in L^2$. By approximation via smooth functions,

$$\nabla f_s = \nabla (f|f|^{s-1}) = s|f|^{s-1}\nabla f$$

so $L^2$-Sobolev gives

$$\| f \|_{qs}^s = \| f|f|^{s-1} \|_q \leq \tilde{c} \left( \| f \|_{2s}^s + \| \nabla (f|f|^{s-1}) \|_2 \right)$$

Now

$$\| \nabla (f|f|^{s-1}) \|_2^2 = s^2 \int |f|^{2s-2} |\nabla f|^2 d\lambda$$

and integration by parts shows

$$\int |f|^{2s-2} |\nabla f|^2 d\lambda = -(2s-2) \int |f|^{2s-2} |\nabla f|^2 d\lambda - \int |f|^{2s-1} \Delta f d\lambda$$

Use PDE $\Delta f = -\beta f|f|^{r-2}$ to wrap this into . . .
Proof of Moser’s iterative estimate continued . . .

\[
\| \nabla (f|f|^{s-1}) \|_2^2 = \frac{\beta s^2}{2s-1} \|f\|_2^{2s+r-2}
\]

Summarizing:

\[
\|f\|_{qs}^s \leq \tilde{c} \left( \|f\|_{2s}^s + \left( \frac{\beta s^2}{2s-1} \right)^{1/2} \|f\|_{2s+r-2}^{s+(r-2)/2} \right).
\]

Assuming \( r \leq 2 \) for simplicity, Hölder gives

\[
\|f\|_{2s+r-2}^{s+(r-2)/2} \leq \lambda(O) \frac{s}{2s+r-2}^{\frac{s}{2s}} \|f\|_{2s}^s
\]

From here we get the basic iterative estimate

\[
\forall s \geq 1: \quad f \in L^{2s} \Rightarrow \|f\|_{qs} \leq \left( \tilde{c} (1 + \tilde{c'} \sqrt{s}) \right)^{1/s} \|f\|_{2s},
\]

where \( \tilde{c'} := \sqrt{\beta} \lambda(O)^{1/2} \).
We now iterate: Set $s_n := (q/2)^n$. Then $f \in L^{2s_0}$ by assumption and so so $f \in L^{2s_n}$ for each $n \geq 0$ with

$$
\|f\|_{2s_n} \leq \left( \prod_{k=0}^{n-1} \left( \tilde{c} \left( 1 + \tilde{c}' \sqrt{s_k} \right) \right)^{1/s_k} \right) \|f\|_2.
$$

The product converges as $n \to \infty$ so

$$
\|f\|_{\infty} = \lim_{n \to \infty} \|f\|_{2s_n} \leq \left( \prod_{k=0}^{\infty} \left( \tilde{c} \left( 1 + \tilde{c}' \sqrt{s_k} \right) \right)^{1/s_k} \right) \|f\|_2
$$

The argument for $r \geq 2$ similar; just observe $qs > 2s + r - 2$ as implied by $q > r$. 

\[\square\]
Key ingredients for Moser iteration:
  • a higher norm of $f$ related to a lower norm of $f$ and $\nabla f$
  • a PDE that converts the norm of $\nabla f$ to a (lower) norm of $f$.
Applies to elliptic PDEs (eigenvalue problems, etc), parabolic PDEs (heat equation etc) including non-linear ones.
To appreciate above derivation, note that any solution $f \in W^{1,2}(O)$ of the Poisson equation

$$-\Delta f = g \quad \text{on } O$$

admits integral representation

$$f = \int_O K(x, y)g(y)\,dy,$$

where $K$ is the Green function in $O$. Smooth away from diagonal $\{(x, x): x \in O\}$ with power law singularity

$$K(x, y) \sim \frac{c}{|x - y|^{d-2}}$$

when $d \geq 3$ and logarithmic singularity in $d = 2$. Key point: convergence requires $g \in L^{d/2+\epsilon}$ for some $\epsilon > 0$!
Back to our variational problem

For us

\[ g := \beta f |f|^r - 2 \]

and, by Moser iteration, \( g \in L^\infty \) (for \( r \in [2, \frac{2d}{d-2}) \)). Continuity+integrability of \( K \implies f \) is continuous.

Since \( y \mapsto \nabla_y K(x, y) \) remains integrable, we even get \( \nabla f \in C^{1,\alpha} \) for all \( \alpha \in (0, 1) \).

Further iteration possible:

\[ \nabla_y K(x, y) = \nabla_x K(x, y) + O(1) \]

with normal derivative of \( K(x, \cdot) \) vanishing on \( \partial O \). So for \( r = 2 \), integrating by parts shows that from \( \nabla f \in C(O) \) we get \( \nabla^2 f \in C(O) \) etc. This ultimately gives \( f \in C^\infty(O) \).
The above set of ideas was developed in the solution of 19th Hilbert problem that asked: Are all the minimizers of

$$\Phi(f) := \int_0 \varphi(\nabla f) d\lambda$$

with $\varphi$ strictly convex and smooth, necessarily smooth?

Solved by E. De Giorgi in 1957, similar ideas used also by J. Nash for parabolic equations. J. Moser in 1960 came up with “Moser iteration” technique used above.
The Sobolev inequality comes along with three other types of inequalities:

- Poincaré inequality
- Nash inequality
- Log-Sobolev inequality

which we will now discuss or at least review.
Lemma ($L^1$-Poincaré inequality)

For each $d \geq 2$,

$$\forall f \in C_c^\infty(\mathbb{R}^d) : \quad \|f\|_1 \leq c(d) \lambda(\text{supp}(f))^{1/d} \|\nabla f\|_1,$$

where $c(d)$ is the constant from $L^1$-Sobolev inequality.

Proof: Hölder’s inequality with parameters $d$ and $\frac{d}{d-1}$ shows

$$\|f\|_1 = \int 1_{\text{supp}(f)} |f| \, d\lambda \leq \lambda(\text{supp}(f))^{1/d} \|f\|_{\frac{d}{d-1}}.$$

Now apply $L^1$-Sobolev inequality.  \qed
Just as for Sobolev, applying this to power function and taking extensions we get:

**Proposition**

For all $d \geq 2$ and let $O \subseteq \mathbb{R}^d$ be non-empty and open with $\lambda(O) < \infty$. Then for all $p \in [1, \infty)$,

$$\forall f \in W_{0}^{1,p}(O) : \quad \|f\|_p \leq p \, c(d) \lambda(O)^{1/d} \|\nabla f\|_p$$

where $c(d)$ is the constant from $L^1$-Sobolev inequality.

Note: Fails for $f \in W^{1,p}(O)$ in general! E.g., $f := 1_O$ has $\nabla f = 0$!
Proof of Proposition

Let $f \in C_c^\infty(O)$, set $g := f|f|^{p-1}$ and note that $\nabla g = pf|f|^{p-2}\nabla f$. Then

$$\|\nabla g\|_1 = p \int |f|^{p-1}|\nabla f| d\lambda \leq pf\|f\|_p^{p-1}\|\nabla f\|_p$$

by Hölder’s inequality. Applying the $L^1$-Poincaré inequality along with $\text{supp}(f) \subseteq O$,

$$\|f\|_p^p = \|g\|_1 \leq c(d)\lambda(O)^{1/d}\|f\|_p^{p-1}\|\nabla f\|_p$$

Now cancel $\|f\|_p^{p-1}$ and apply density of $C_c^\infty(O)$ in $W_0^{1,p}(O)$. $\square$
Corollary

Let $d \geq 2$ and let $O \subseteq \mathbb{R}^d$ be non-empty open with $\lambda(O) < \infty$. For $p \in [1, \infty)$ denote $p^{**} := \frac{dp}{d-p}$ if $p < d$ and $p^{**} := \infty$ if $p \geq d$. Then for each $q \in (0, p^{**})$ there is $c(d, p, q, O) \in (0, \infty)$ such that

$$\forall f \in W_0^{1,p}(O) : \quad \|f\|_q \leq c(d, p, q, O)\|\nabla f\|_p.$$  

Proof: For $p > d$ this is by Morrey’s inequality. For $p < d$ and $q \in [p, p^*)$, we interpolate $q$-norm into $p$-norm and $p^*$-norm and then apply Sobolev to $p^*$ norm and Poincaré to $p$-norm. The case $p = d$ handled by a perturbation argument.
Extension to $W^{1, p}(O)$ possible assuming $O$ has a nice boundary. The key underlying observation is:

Lemma

Let $d \geq 2$ and let $O \subseteq \mathbb{R}^d$ be non-empty, open with $\lambda(O) < \infty$. Then for all $p \in [1, \infty)$ and all $f \in C_c^\infty(O)$,

$$\left( \int_O \left| f - \frac{1}{\lambda(O)} \int_O f \, d\lambda \right|^p \, d\lambda \right)^{1/p} \leq 2p c(d) \lambda(O)^{1/d} \| \nabla f \|_p.$$

Proof: Denote $\tilde{f} := 1_O \frac{1}{\lambda(O)} \int_O f \, d\lambda$. Then LHS bounded by

$$\|f\|_p + \|\tilde{f}\|_p \leq 2\|f\|_p$$

where second $\leq$ comes from Jensen's inequality. Now apply $L^p$-Poincaré. \qed
Lemma (Nash inequality)
For each $d \geq 2$ there is $c \in (0, \infty)$ such that

$$\forall f \in W^{1,2}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d): \quad \|f\|_2^{1+2/d} \leq c \|
abla f\|_2 \|f\|_1^{2/d}$$

Key in analyzing the heat flow which preserves the $L^1$-norm!
Problem with Sobolev: \( \frac{dp}{d-p} \rightarrow p \) as \( d \rightarrow \infty \), so no estimate uniform in \( d \geq 2 \) possible.
For analysis in infinite dimensions, we instead use:

Lemma (Log-Sobolev inequality)
For all \( d \geq 1 \) there is \( c \in (0, \infty) \) such that

\[
\forall f \in W^{1,1}(\mathbb{R}^d): \quad \int |f| \log\left(\frac{|f|}{\|f\|_1}\right) \leq c \|\nabla f\|_1
\]

P. Federbush (1969) extended this to \( \mathbb{R}^d \) based on hypercontractivity estimates of Nelson. Full power realized by L. Gross.
Classical treatment of Fourier analysis
Analysis and functional analysis on $C(X)$ — Arzelà-Ascoli, Stone-Weierstrass and Riesz-Markov representation theorem
Theory of distributions
Abstract functional analysis.

Well, some other time ...
THANK YOU
Stay well & safe!