Calderón-Zygmund theory

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Outline:

- Statement and motivation
- Proof via Marcinkiewicz and duality
- Applications to Hilbert and inverse-Fourier transform
- General Calderón-Zygmund kernels
Q: What happens with Riesz transform

\[ T_\alpha f(x) := \int \frac{1}{|x - y|^\alpha} f(y) \, dy \]

when \( \alpha = d \)? Or with Hilbert transform?

A: Integral not defined even for nice \( f \) due to singularity at \( x = y \), but could truncate to \( |x - y| \geq \epsilon \), perhaps.

Singularity as \( |x| \to \infty \) bad too; kernel

\[ K(x) := \frac{1}{|x|^d} 1_{|x| \geq \epsilon} \]

obeys \( K \in L^{1,w} \), but Schur’s test requires \( L^{r,w} \) with \( r > 1 \).
Theorem (Calderón-Zygmund)

Consider the measure space \((\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d), \lambda)\) for \(d \geq 1\). For all \(A, B > 0\), all \(M > 1\) and all \(p \in (1, \infty)\) there is \(C_p \in (0, \infty)\) such that for all measurable kernels \(K: \mathbb{R}^d \to \mathbb{R}\) satisfying

\[
K \in L^2 \text{ with Fourier transform } \hat{K} \text{ obeying } \|\hat{K}\|_{\infty} \leq A
\]

and

\[
\sup_{z \in \mathbb{R}^d \setminus \{0\}} \int_{|x| > M|z|} |K(x - z) - K(x)| \, dx \leq B,
\]

the convolution operator \(T_Kf := K \ast f\) is well defined by the integral expression for all \(f \in L^1\) and extends continuously to a map \(L^p \to L^p\) for each \(p \in (1, \infty)\) with

\[
\forall p \in (1, \infty): \quad \|T_K\|_{L^p \to L^p} \leq C_p
\]
1st condition ensures $K$ is locally integrable and $K \ast f$ meaningful for $f \in L^1$ (by Young convolution inequality)

2nd condition often stated as $K \in C^1(\mathbb{R}^d \setminus \{0\})$ with

$$
\forall x \in \mathbb{R}^d \setminus \{0\} : \quad |\nabla K(x)| \leq \frac{\tilde{B}}{|x|^{d+1}}
$$

which (along with 1st condition) gives

$$
|K(x)| \leq \frac{\tilde{B}/d}{|x|^d}
$$

and so we cannot hope for more than $K \in L^{1,w}$ (and so Schur’s test is still out).

Upshot: Trading local regularity against integrability
1st condition implies $T_K$ is strong type $(2, 2)$

with 2nd condition this implies $T_K$ is weak type $(1, 1)$. This is the key novelty; requires so called Calderon-Zygmund decomposition of $\mathbb{R}^d$ into sets where $f$ is bounded and sets where $f$ has bounded integral

Marcinkiewicz interpolation gives

$$T_K \text{ is strong type } (p, p) \text{ for all } p \in (1, 2]$$

Duality: true also for $p \in [2, \infty)$
Improved Young convolution inequality

Recall: By Young convolution inequality \( f \in L^2 \) and \( g \in L^1 \) implies integral \( f \star g \) converges absolutely a.e. and

\[
\|f \star g\|_2 \leq \|f\|_2 \|g\|_1
\]

Need a slight improvement:

Lemma

\[
\forall f \in L^1 \ \forall g \in L^2 : \quad \|f \star g\|_2 \leq \|f\|_2 \|\hat{g}\|_\infty
\]

Proof: Let \( f \in L^1 \) and \( g \in L^2 \). Fourier transform isometry so

\[
\|\hat{f} \star \hat{g}\|_2 \leq \|f\|_1 \|g\|_2
\]

Hence \( g \mapsto \hat{f} \star \hat{g} \) continuous. If \( g \in L^1 \), then \( \hat{f} \star \hat{g} = \hat{fg} \) so true for \( g \in L^2 \) as well. Hence

\[
\|f \star g\|_2 = \|\hat{f} \star \hat{g}\|_2 = \|\hat{fg}\|_2 \leq \|f\|_2 \|\hat{g}\|_\infty = \|f\|_2 \|\hat{g}\|_\infty \quad \square
\]
Corollary

The operator $T_K$ is strong type $(2, 2)$ with $\|T_K\|_{L^2 \to L^2} \leq A$

Proof: For $f \in L^1$, $T_Kf$ well defined via $K \ast f$ and obeys

$$\|T_Kf\|_2 \leq A\|f\|_2$$

by above lemma. So $T_K$ extends to $L^2$ with $\|T_K\|_{L^2 \to L^2} \leq A$ \qed
Dyadic cube is any cube of the form
\[ 2^n x + [0, 2^n)^d \]
for \( x \in \mathbb{Z}^d \) and \( n \in \mathbb{Z} \).

Lemma (Calderón-Zygmund decomposition)

Let \( f \in L^1 \) and \( t > 0 \). Then there exist disjoint dyadic cubes \( \{Q_i\}_{i \in I} \) such that
\[
\forall i \in I: \quad t \lambda(Q_i) < \int_{Q_i} |f| \, d\lambda \leq 2^d t \lambda(Q_i)
\]
and
\[
|f| \leq t \quad \lambda\text{-a.e. on } \mathbb{R}^d \setminus \bigcup_{i \in I} Q_i
\]
Call a dyadic cube $Q$ **good** if

$$\frac{1}{\lambda(Q)} \int_Q |f| \, d\lambda \leq t$$

and call it **bad** otherwise. For $n \in \mathbb{Z}$ such that $\int |f| \, d\lambda \leq t2^n$, all dyadic cubes of side-length $2^n$ are good. Let $\{Q_i\}_{i \in I}$ enumerate the set of all bad dyadic cubes $Q$ such that the (unique) dyadic cube $Q'$ containing $Q$ and having side length twice as that of $Q$ is good. Then

$$t\lambda(Q) < \int_Q |f| \, d\lambda \leq \int_{Q'} |f| \, d\lambda \leq t\lambda(Q') = 2^d t\lambda(Q)$$

because $Q$ is bad and $Q'$ is good.

If $x$ lies only in good cubes, Lebesgue differentiation shows $|f(x)| \leq t$ a.e. (Need a version for dyadic cubes; proved when discussed martingale convergence.)
Proposition

$T_K$ is weak type $(1,1)$. Explicitly,

$$\exists c \in (0, \infty) \ \forall f \in L^1 \ \forall t > 0: \ \lambda(|T_Kf| > t) \leq \frac{c}{t} \|f\|_1$$

where $c$ depends only on $d$ and the constants $A$ and $B$
Decomposition of $f$

Pick $f \in L^1$ and $t > 0$ and let $\{Q_i\}_{i \in I}$ be as above. Set

$$F := \mathbb{R}^d \setminus \bigcup_{i \in I} Q_i$$

define $g : \mathbb{R}^d \to \mathbb{R}$ by

$$g(x) := \begin{cases} \frac{1}{\lambda(Q_i)} \int_{Q_i} f \, d\lambda & \text{if } x \in Q_i \text{ for some } i \in I \\ f(x) & \text{if } x \in F \end{cases}$$

and abbreviate

$$h(x) := f(x) - g(x)$$

Note that

$$h = 0 \text{ on } F \quad \land \quad \forall i \in I : \int_{Q_i} h \, d\lambda = 0$$

Union bound + additivity:

$$\lambda(|T_K f| > t) \leq \lambda(|T_K g| > t/2) + \lambda(|T_K h| > t/2)$$

Now estimate each term separately …
Tails of $T_{Kg}$

Will use that $T_K$ maps $L^2 \to L^2$ with norm $\leq A$ (proved above). Need to estimate

$$\|g\|_2^2 = \int_F g^2 \, d\lambda + \sum_{i \in I} \int_{Q_i} g^2 \, d\lambda$$

$$\leq \int_F |f| \, d\lambda + \sum_{i \in I} \left( \frac{1}{\lambda(Q_i)} \int_{Q_i} f \, d\lambda \right)^2 \lambda(Q_i)$$

$$\leq \int_F |f| \, d\lambda + \sum_{i \in I} (2^d t)^2 \lambda(Q_i)$$

$$\leq t \int_F |f| \, d\lambda + 4^d t \sum_{i \in I} \int_{Q_i} |f| \, d\lambda$$

$$= t \int_F |f| \, d\lambda + 4^d t \int_{F^c} |f| \, d\lambda = (4^d + 1) t \|f\|_1$$

Hence

$$\lambda(|T_{Kg}| > t/2) \leq \frac{4}{t^2} \|T_{Kg}\|_2^2 \leq \frac{4A^2}{t^2} \|g\|_2^2 \leq \frac{4A^2(4^d + 1)}{t} \|f\|_1$$
Consider $h_i := h_{1_{Q_i}}$. Let $y_i :=$ the center of $Q_i$. As $\int_{Q_i} h \, d\lambda = 0$,

$$T_K h_i(x) = \int_{Q_i} K(x - y) h_i(y) \, dy = \int_{Q_i} (K(x - y) - K(x - y_i)) h_i(y) \, dy$$

Let $Q'_i :=$ the cube of $M\sqrt{d}$-times the side length of $Q_i$ centered at $y_i$. By Tonelli and 2nd condition:

$$\int_{\mathbb{R}^d \setminus Q'_i} |T_K h_i| \, d\lambda$$

$$\leq \int_{Q_i} \left( \int_{\mathbb{R}^d \setminus Q'_i} |K(x - y_i + y - y_i) - K(x - y_i)| \, dx \right) |h_i(y)| \, dy$$

$$\leq B \int_{Q_i} |h| \, d\lambda \leq 2B \int_{Q_i} |f| \, d\lambda$$

which uses $|x - y_i| > M|y - y_i|$ for all $x \notin Q'_i$ and all $y \in Q_i$. Then ...

Tails of $T_Kh$ continued …

… abbreviating $F' := \mathbb{R}^d \setminus \bigcup_{i \geq 1} Q'_i$ we thus get

$$
\int_{F'} |T_Kh| \, d\lambda \leq 2B \|f\|_1
$$

On the other hand,

$$
\lambda(\mathbb{R}^d \setminus F') \leq \sum_{i \in I} \lambda(Q'_i) = (M \sqrt{d})^d \sum_{i \in I} \lambda(Q_i)
$$

$$
\leq \frac{(M \sqrt{d})^d}{t} \sum_{i \in I} \int_{Q_i} |f| \, d\lambda \leq \frac{(M \sqrt{d})^d}{t} \|f\|_1
$$

and so

$$
\lambda(|T_Kh| > t/2) \leq \lambda(\mathbb{R}^d \setminus F') + \frac{2}{t} \int_{F'} |T_Kh| \, d\lambda \leq \frac{(M \sqrt{d})^d + 4B}{t} \|f\|_1
$$

So claim holds with

$$
c := 4A^2(4^d + 1) + (M \sqrt{d})^d + 4B \quad \square
$$
Marcinkiwicz: $T_K$ strong type $(p, p)$ for $p \in (1, 2]$. Now let $q \in (2, \infty)$ and let $p$ be such that $p^{-1} + q^{-1} = 1$. Then duality between $L^p$ and $L^q$ gives

\[
\forall f \in L^1 \cap L^p \ \forall g \in L^q : \quad \left| \int g(K \ast f) \, d\lambda \right| \leq \|T_K\|_{L^p \to L^p} \|f\|_p \|g\|_q
\]

For $f \in L^1$ integral $K \ast f$ converges absolutely. So by Fubini-Tonelli:

\[
\forall f \in L^1 \cap L^p \ \forall g \in L^q \cap L^1 : \quad \left| \int (T_K g) f \, d\lambda \right| \leq \|T_K\|_{L^p \to L^p} \|f\|_p \|g\|_q
\]

Density of $L^p \cap L^1$ in $L^p$ gives

\[
\forall g \in L^q \cap L^1 : \quad \|T_K g\|_q \leq \|T_K\|_{L^p \to L^p} \|g\|_q.
\]

which implies that $T_K$ extends continuously to a map $L^q \to L^q$ with

\[
\|T\|_{L^q \to L^q} \leq \|T_K\|_{L^p \to L^p}
\]

(Equality holds by duality.)
Application to Hilbert transform

Recall: $Hf$ defined as the $\epsilon \downarrow 0$ limit of convolution-type operator $H_\epsilon f := K_\epsilon \ast f$ where

$$K_\epsilon(x) := \frac{1}{\pi x} \mathbf{1}_{(\epsilon, 1/\epsilon)}(|x|)$$

Convergence pointwise for $f \in C^1(\mathbb{R}) \cap L^1$ and in $L^2$ for $f \in L^2$

Theorem (Hilbert transform in $L^p$)

We have

$$\forall p \in (1, \infty) : \sup_{0 < \epsilon < 1} \|H_\epsilon\|_{L^p \to L^p} < \infty.$$ 

In particular, for all $p \in (1, \infty)$, there exists a continuous linear operator $H : L^p \to L^p$ such that

$$\forall f \in L^p : \quad H_\epsilon f \quad \underset{\epsilon \downarrow 0}{\longrightarrow} \quad Hf \quad \text{in} \quad L^p.$$
Proof of Theorem

For strong type $(2, 2)$, use Fourier calculation to get

$$\hat{K}_\epsilon(z) = -\frac{2i}{\pi} \int_{\epsilon < t < 1/\epsilon} \frac{\sin(2\pi z t)}{t} \, dt$$

Hence, $A := \sup_{0 < \epsilon < 1} \|\hat{K}_\epsilon\|_\infty < \infty$.

For weak type $(1, 1)$, compute

$$|K_\epsilon(x - z) - K_\epsilon(x)|$$

$$\leq \left| \frac{1}{x - z} - \frac{1}{x} \right| + \frac{1}{|x|} |1_{(\epsilon, 1/\epsilon)}(|x - z|) - 1_{(\epsilon, 1/\epsilon)}(|x|)|$$

$$\leq \frac{2|z|}{|x|^2} + \frac{1}{|x|} 1_{\{(1/\epsilon - |z|, 1/\epsilon + |z|) \}}(|x|) + \frac{1}{|x|} 1_{\{(\epsilon - |z|, \epsilon + |z|) \}}(|x|)$$

Need to integrate this over $|x| > 2|z|$. First term easy. For the other two terms we note that ...
Proof of Theorem continued . . .

...for any $a, b > 0$ with $\max\{2b, a - b\} < a + b$,

$$\int_{\max\{2b,a-b\}}^{a+b} \frac{dx}{x} = \log\left(\frac{a+b}{\max\{2b,a-b\}}\right)$$

Examining $a - b < 2b$ and $a - b > 2b$ separately, RHS $\leq \log(2)$. Now use this with $a := \epsilon, 1/\epsilon$ and $b = |z|$ to get

$$B := \sup_{0<\epsilon<1} \sup_{z \in \mathbb{R} \setminus \{0\}} \int_{|x|>2|z|} |K_\epsilon(x - z) - K_\epsilon(x)| \, dx < \infty$$

So $\{H_\epsilon\}_{0<\epsilon<1}$ obey conditions of Calderón-Zygmund theorem with uniform $A$ and $B$ (and $M := 2$). So we get

$$\sup_{0<\epsilon<1} \|H_\epsilon\|_{L^p \to L^p} < \infty.$$ 

It remains to address convergence $H_\epsilon f \to Hf$ . . .
...which we already know in $L^2$ by Fourier techniques. We will use interpolation for $L^p$-norms.

Given $p \in (1, \infty)$, choose $\tilde{p} \in (1, p)$ when $p < 2$ or $\tilde{p} \in (p, \infty)$ when $p > 2$. Then $\frac{1}{p} = (1 - \theta) \frac{1}{\tilde{p}} + \theta \frac{1}{2}$ for some $\theta \in (0, 1)$ and so

$$\forall f \in \tilde{L}^p \cap L^2: \quad \|H_\epsilon f - H_\delta f\|_p \leq \|H_\epsilon f - H_\delta f\|_2^\theta \|H_\epsilon f - H_\delta f\|_{\tilde{p}}^{1-\theta}.$$ 

Now $\|H_\epsilon f - H_\delta f\|_2 \to 0$ as $\epsilon, \delta \downarrow 0$ by the claim in $L^2$ while

$$\|H_\epsilon f - H_\delta f\|_{\tilde{p}} \leq \left(2 \sup_{0 < \epsilon' < 1} \|H_\epsilon\|_{L^\tilde{p} \to L^\tilde{p}}\right) \|f\|_{\tilde{p}}.$$ 

Completeness of $L^p$ shows $H_\epsilon f \to Hf$ for each $f \in \tilde{L}^p \cap L^2$. As $\tilde{L}^p \cap L^2$ dense in $L^p$, true for all $f \in L^p$. \qed
Q: Is $H_\epsilon f \to Hf$ uniform in $f \in L^p$ with $\|f\|_p \leq 1$?

A: Not in $L^2$ (and by duality in interpolation, not in $L^p$) because

$$\|H_\epsilon - H\|_{L^2 \to L^2} = \|\hat{K}_\epsilon - \hat{K}\|_{\infty}$$

and RHS does not tend to zero because $K_\epsilon$ is continuous and

$\hat{K}(z) := (-i)\text{sgn}(z)$ is not.
Definition (Strong and norm convergence of operators)

A sequence \( \{T_n\}_{n \geq 1} \) of linear operators on a normed linear space \( V \) is said to converge strongly to a linear operator \( T \) if

\[
\forall f \in V : \lim_{n \to \infty} \|T_n f - T f\| = 0
\]

The sequence \( \{T_n\}_{n \geq 1} \) converges to \( T \) in (operator) norm if

\[
\lim_{n \to \infty} \|T_n - T\| = 0
\]

So, on \( L^p \) with \( p \in (1, \infty) \), we get \( H \epsilon \to H \) strongly but not in operator norm.
Cotlar’s approach

Cotlar’s identity:

\[(Hf)^2 = f^2 + H(f(Hf))\]

By induction: For \(n \geq 1\) and \(p := 2^n\),

\[\|Hf\|_{2p}^2 \leq p\|f\|_{2p}^2 + p\|H(f(Hf))\|_p^p\]

\[\leq p\|f\|_{2p}^2 + p(\|H\|_{Lp \to Lp})^p\|f\|_{2p}^p\|Hf\|_{2p}^p\]

Proves \(\|H\|_{Lp \to Lp} < \infty\) for \(p \in \{2^n : n \geq 1\}\). Interpolation + duality gives this for all \(p \in (1, \infty)\).
Partial Fourier inversions

For $f \in L^1$, define

$$T_n f (x) := \int_{-n}^{n} \hat{f}(k) e^{-2\pi i k \cdot x} \, dk,$$

where $\hat{f} :=$ Fourier transform of $f$. Know that, if $\hat{f} \in L^1$, then $T_n f \rightarrow f$ pointwise. Q: Convergence in $L^p$?

Theorem

Let $p \in (1, \infty)$. Then, for each $n \geq 1$, the operator $T_n$ extends continuously to a map $L^p \rightarrow L^p$ and

$$\forall f \in L^p : \quad \lim_{n \rightarrow \infty} T_n f \quad \text{in} \quad L^p$$

Proof: homework
A.e. convergence

$L^p$ convergence gives a.e. convergence along a subsequence.

Need for subsequences removed by L. Carleson (1966) for $L^2$ for $L^2$ and by R. Hunt (1968) for $L^p$ (1 < $p$ < $\infty$). Key idea: Carleson operator

$$T*f(x) := \sup_{n \geq 1} \left| \int_{-n}^{n} \hat{f}(k)e^{-2\pi ik \cdot x} dk \right|,$$

is weak type $(2,2)$.


No a.e. convergence for $L^1$ functions (A.N. Kolmogorov’s counterexample)
Definition (Calderón-Zygmund type)

Given $A, B > 0$ and $M > 1$, a linear operator $T: C_c(\mathbb{R}^d) \to L^0$ we say that $T$ is Calderón-Zygmund type with parameters $(A, B, M)$ if

$$\forall f \in C_c(\mathbb{R}^d): \quad \|Tf\|_2 \leq A\|f\|_2$$

and there is a measurable kernel $K: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that

$$\sup_{y \in \mathbb{R}^d} \sup_{z \in \mathbb{R}^d \setminus \{0\}} \int_{|x-y| > M|z|} |K(x, y + z) - K(x, y)| \, dx \leq B$$

for which $T$ admits the integral representation

$$\forall f \in C_c(\mathbb{R}^d): \quad Tf(\cdot) = \int K(\cdot, y)f(y) \, dy \quad \lambda\text{-a.e.}$$

with integral absolutely convergent $\lambda$-a.e.
Note: Assuming strong type \((2, 2)\)! Sufficient conditions exist:

**Lemma**

Let \((X, \mathcal{F}, \mu)\) and \((Y, \mathcal{G}, \nu)\) be \(\sigma\)-finite measure spaces and let \(K: X \times Y \to \mathbb{R}\) be \(\mathcal{F} \otimes \mathcal{G}\)-measurable and such that \(K \in L^2(\mu \otimes \nu)\). Then for each \(f \in L^2\), the integral in

\[
Tf(x) := \int K(x, y)f(y)\nu(dy)
\]

converges absolutely for \(\nu\)-a.e. \(x \in X\) and defines a continuous linear operator \(L^2(\nu) \to L^2(\mu)\). Moreover,

\[
\|T\|_{L^2(\nu) \to L^2(\mu)} \leq \|K\|_{L^2(\mu \otimes \nu)}
\]

These are usually too weak to be used here. Strong type \((2, 2)\) property usually verified by “Hilbert space” techniques.
Theorem (Calderón-Zygmund, general form)

For all $A, B > 0, M > 1, d \geq 1$ and $p \in [1, 2]$ there is $C_p \in (0, \infty)$ such that for every Calderón-Zygmund-type operator $T$ with parameters $(A, B, M)$, we have:

1. $T$ is weak type $(1, 1)$ with (its extension to $L^1$ satisfying)

   $$\forall t > 0 \forall f \in L^1 : \quad \lambda(|Tf| > t) \leq \frac{C_1}{t} \|f\|_1$$

2. For each $p \in (1, 2]$, $T$ is strong type $(p, p)$ with (its extension to $L^p$ satisfying)

   $$\forall f \in L^p : \quad \|T\|_p \leq C_p \|f\|_p$$

If $K^*(x, y) := K(y, x)$ is also C.Z.-type with the same $A, B, M$, then $T$ is also strong type $(p, p)$ for every $p \in [2, \infty)$ with $C_p := C \frac{p}{p-1}$.
Proof: main changes

The proof for $p \in (1, 2]$ is taken nearly verbatim. Duality argument requires some work. First some functional analysis:

Lemma

Let $V$ be a normed linear space and let $T: \text{Dom}(T) \to V$ be a linear operator on $V$ with dense linear $\text{Dom}(T)$. For each $\phi \in V^*$,

$$\forall f \in \text{Dom}(T): \quad (T^* \phi)(f) := \phi(Tf)$$

defines a linear functional $T^* \phi$ on $\text{Dom}(T)$. The map $\phi \mapsto T^* \phi$ is linear and so $T^*$ is a linear operator called the adjoint of $T$. If $T$ is bounded, then $T^* \phi \in V^*$ and $T^*$ extends to a continuous linear operator $T^*: V^* \to V^*$ with

$$\|T^*\| \leq \|T\|.$$ 

(Equality holds by the Hahn-Banach theorem.)
Proof of Lemma

Linearity of $T^*$ clear. For $T$ bounded,

$$\forall f \in \text{Dom}(T) \ \forall \phi \in \mathcal{V}^* : \ \left| (T^* \phi)(f) \right| \leq \|\phi\| \|Tf\| \leq \|T\| \|\phi\| \|f\|$$

As $\text{Dom}(T)$ is dense in $\mathcal{V}$, $T^* \phi$ extends continuously to $\mathcal{V}$ with

$$\|T^* \phi\| \leq \|T\| \|\phi\|$$

Hence $\|T^*\| \leq \|T\|$.

□
Lemma

Let $p, q \in (1, \infty)$ be such that $p^{-1} + q^{-1} = 1$ and let $T : C_c(\mathbb{R}) \rightarrow L^0$ be a linear operator such that

$$\forall f \in C_c(\mathbb{R}^d) : \quad Tf(\cdot) = \int K(\cdot, y)f(y)dy \quad \lambda\text{-a.e.}$$

with the integral convergent $\lambda\text{-a.e.}$ If $T$ is continuous as a map $L^p \rightarrow L^p$, its adjoint $T^*$ admits the integral representation

$$\forall f \in C_c(\mathbb{R}^d) : \quad T^*f(\cdot) = \int K(y, \cdot)f(y)dy \quad \lambda\text{-a.e.}$$

where the integral converges $\lambda\text{-a.e.}$.
For $f, g \in C_c(\mathbb{R}^d)$, Fubini-Tonelli gives

$$\int g(Tf) \, d\lambda = \int g(x) \left( \int_{|x-y|>\epsilon} K(x,y)f(y) \, dy \right) \, dx$$

$$= \int f(y) \left( \int K(x,y)g(x) \, dx \right) \, dy = \int f(\tilde{T}g) \, d\lambda$$

where $\tilde{T}g := \int K(y, \cdot) f(y) \, dy$ converges absolutely.

Riesz representation:

$$\phi_g(Tf) = \phi_{\tilde{T}g}(f)$$

Using that $g \mapsto \phi_g$ is bijective isometry of $(L^p)^* \to L^q$, we now identify $T^*\phi_g$ with $\tilde{T}g$. \qed
Pick $p \in (2, \infty)$ and let $q$ be Hölder dual. Let $\tilde{T}$ be defined using $K^*$-kernel (which is C.Z.-type). Then Lemma says

$$\tilde{T}^* = T$$  \hspace{1cm} (1)

and so

$$\|T\|_{L^p \to L^p} \leq \|\tilde{T}\|_{L^q \to L^q} \leq C_q \quad \square$$  \hspace{1cm} (2)