24. MARCINKIEWICZ INTERPOLATION

We proceed by stating and proving an interpolation theorem due to Marcinkiewicz and then discuss some applications.

24.1 Diagonal Marcinkiewicz interpolation.

Let us start with a bit of history. In 1937 J. Marcinkiewicz, a young and prolific Polish mathematician spending time on scholarships in England, announced a new interpolation result in “Sur l’interpolation d’operations” (C. R. Acad. Sci. Paris, 208, 1272–1273). When Germany and Soviet Union invaded Poland in 1939, Marcinkiwicz quickly returned to Poland and enlisted in the Polish army in Vilnius (Lithuania). Being taken a prisoner of war by the Soviets, he is believed to have died in the 1940 Katyn massacre of Polish officers orchestrated by the Soviets.

In 1956, A. Zygmund, by then a famous harmonic analyst of Polish origin who was Marcinkiwicz’ advisor and collaborator in Vilnius before the war and who worked in the US since 1940, recreated Marcinkiewicz’ proof based on a letter he had received from him before the war. The result was ultimately published in “On a theorem of Marcinkiewicz concerning interpolation of operations” (J. Math. Pures Appl., 35: 223–248) and is one of the cornerstones of modern harmonic analysis.

We begin by noting a simple interpolation fact for the quasinorms:

**Lemma 24.1** (Interpolation for quasinorms) For $p_0, p_1 \in (0, \infty]$ and $\theta \in [0, 1]$, define $p$ by

\[
\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.
\]

Then

\[
\forall f \in L^p: \quad \|f\|_p \leq \|f\|_{p_0}^{1-\theta} \|f\|_{p_1}^\theta
\]

Moreover, if $p_0 < p_1$, then for all $\theta \in (0, 1)$,

\[
\forall f \in L^p: \quad \|f\|_p \leq \left[ \frac{p}{p-p_0} + \frac{p}{p_1-p} \right]^{1/p} \|f\|_{p_0}^{1-\theta} \|f\|_{p_1}^\theta
\]

**Proof.** Assuming $f \neq 0$ and $p_0, p_1 < \infty$, for each $t > 0$ we have

\[
t \mu(|f| > t) \leq \left( t \mu(|f| > t) \right)^{1/p_0} \left( t \mu(|f| > t) \right)^{1/p_1} \leq \|f\|_{p_0}^{1-\theta} \|f\|_{p_1}^\theta.
\]

Taking supremum over $t > 0$ gives (24.2). The case when $p_0 < p_1 = \infty$ follows by noting that then $1/p_0 = (1-\theta)/p_0$ and the supremum can be reduced to $t \leq \|f\|_\infty$.

For the second part of the claim, we assume first $p_1 < \infty$ and recall the identity (23.48) to write, for any $a > 0$,

\[
\|f\|_p = \int_0^a p t^{p-1} \mu(|f| > t) \, dt + \int_a^\infty p t^{p-1} \mu(|f| > t) \, dt
\]
Using the definitions of \([f]_{p_0}\) and \([f]_{p_1}\) along with \(p_0 < p < p_1\) this becomes
\[
\|f\|_p^p \leq p[f]_{p_0}^p \int_0^a t^{p-p_0-1} \, dt + p[f]_{p_1}^p \int_a^\infty t^{p-p_1-1} \, dt \\
= \frac{p}{p-p_0} [f]_{p_0}^p a^{p-p_0} + \frac{p}{p_1-p} [f]_{p_1}^p a^{p-p_1}
\] (24.6)

Optimizing over \(a > 0\) we then get (24.3). In the case when \(p_1 = \infty\) we take \(a := \|f\|_\infty\), which eliminates the second integral in (24.5).

Notice that the prefactor in (24.3) diverges as \(p \downarrow 0\) or \(p \uparrow p_1\). This is expected based on the assumptions but should improve when we assume that \(f \in L^{p_0}\) or \(f \in L^{p_1}\). Indeed, if we know \(\|f\|_{p_0} < \infty\), then the first integral in (24.5) can be bounded by \(\|f| \wedge a\|_p^p\) which is in turn dominated by \(\|f|_{p_0}^p a^{p-p_0}\). Using this in (24.6) effectively eliminates the term \(\frac{p}{p-p_0}\) at the cost of replacing \([f]_{p_0}\) by \(\|f\|_{p_0}\). The same occurs in the second term in (24.5) when \(\|f\|_{p_1} < \infty\); we just write the integral as \(\|(\|f| - a) 1_{\{|f| > a\}}\|_p^p\).

We proceed by a simpler version of the interpolation theorem for operators that is already quite useful and demonstrates most of the underlying ideas. First a definition:

**Definition 24.2** An operator \(T : \text{Dom}(T) \to L^0\) on a linear subspace \(\text{Dom}(T) \subseteq L^0\) is said to be sublinear if
\[
\forall f, g \in \text{Dom}(T) : \quad |T(f + g)| \leq |Tf| + |Tg| \tag{24.7}
\]
and
\[
\forall f \in \text{Dom}(T) \forall c \in \mathbb{R} : \quad |T(cf)| = |c||Tf| \tag{24.8}
\]
If the space \(L^0\) is over \(C\), then this holds with \(c \in \mathbb{C}\).

An example of an operator that is sublinear but not linear is the Hardy-Littlewood maximal function. The reason why we include the homogeneity (24.8) is to make the operator norm, as well as comparisons with other norms, meaningful. Sublinear operators behave very similarly to linear operators as far as their continuity properties are concerned. In particular, boundedness is still equivalent to continuity.

**Theorem 24.3** (Marcinkiewicz interpolation theorem, diagonal form) Let \(p_0, p_1 \in (0, \infty]\) obey \(p_0 < p_1\) and let \(T : (L^{p_0} + L^{p_1})(X, \mathcal{F}, \mu) \to L^0(Y, \mathcal{G}, \nu)\) be a sublinear operator such that
\[
\exists C_0 \in (0, \infty) \forall f \in L^{p_0} : \quad [Tf]_{p_0} \leq C_0 \|f\|_{p_0} \tag{24.9}
\]
and
\[
\exists C_1 \in (0, \infty) \forall f \in L^{p_1} : \quad [Tf]_{p_1} \leq C_1 \|f\|_{p_1} \tag{24.10}
\]
Then for all \(p \in (p_0, p_1)\) we have
\[
\forall f \in L^p : \quad \|Tf\|_p \leq 2 \left[ \frac{p}{p-p_0} + \frac{p}{p_1-p} \right]^{\gamma/p} C_0^{1-\theta} C_1^{\theta} \|f\|_p \tag{24.11}
\]
where \(\theta \in (0, 1)\) is the unique number such that (24.1) holds.

**Proof.** The main idea of the proof is very similar to that of Lemma 24.1. Let \((X, \mathcal{F}, \mu)\) be the underlying measure space. Assume first that \(p_1 < \infty\) and let \(p \in (p_0, p_1)\). As
bounded measurable $f : X \to \mathbb{R}$ with $\mu(\text{supp}(f)) < \infty$ are dense in $L^p$, it suffices to
prove the claim for such an $f$. Fix $t > 0$ and $a > 0$ and write
\[ f_0 := f 1_{\{|f| > at\}} \quad \wedge \quad f_1 := f 1_{\{|f| \leq at\}}. \tag{24.12} \]
These functions depend on $t$ without that being explicit in the notation. As $f = f_0 + f_1$, the sublinearity of $T$ and a union bound give
\[ \nu(|Tf| > t) \leq \nu(|Tf_0| > t/2) + \nu(|Tf_1| > t/2). \tag{24.13} \]
The assumptions (24.9–24.10) yield
\[ \nu(|Tf_0| > t/2) \leq C_0^{\frac{2p_0}{tp_0}} \int_{\{|f| > at\}} |f|^{p_0} d\mu \tag{24.14} \]
and
\[ \nu(|Tf_1| > t/2) \leq C_1^{\frac{2p_1}{tp_1}} \int_{\{|f| \leq at\}} |f|^{p_1} d\mu \tag{24.15} \]
Since
\[ \|Tf\|_p^p = \int_0^\infty p t^{p-1} \nu(|Tf| > t) dt \tag{24.16} \]
we need to calculate, using Tonelli’s theorem,
\[ \int_0^\infty t^{p-1} \left( \frac{1}{tp_0} \int_{\{|f| > at\}} |f|^{p_0} d\mu \right) dt = \frac{a^{p_0-p}}{p-p_0} \int |f|^p d\mu \tag{24.17} \]
which uses $p > p_0$, and
\[ \int_0^\infty t^{p-1} \left( \frac{1}{tp_1} \int_{\{|f| \leq at\}} |f|^{p_1} d\mu \right) dt = \frac{a^{p_1-p}}{p_1-p} \int |f|^p d\mu \tag{24.18} \]
which uses $p < p_1$. Inserting (24.14–24.15) in (24.13) and invoking (24.16), the bounds (24.17–24.18) combine into
\[ \|Tf\|_p^p \leq \left[ (2C_0)^{p_0} a^{p_0-p} \frac{p}{p-p_0} + (2C_1)^{p_1} a^{p_1-p} \frac{p}{p_1-p} \right] \|f\|_p^p \tag{24.19} \]
Minimizing over $a > 0$ then gives the claim when $p_1 < \infty$.

For $p_1 = \infty$ we use that $[Tf_1]_\infty = \|Tf_1\|_\infty$ and so (24.10) implies
\[ \|Tf_1\|_\infty \leq C_1 at. \tag{24.20} \]
For $a := \frac{1}{2C_1}$ this gives $\nu(|Tf_1| > t/2) = 0$. This eliminates the second term in the large bracket on the right of (24.19), which can just as well be achieved (for the above choice of $a$) by taking $p_1 \to \infty$ there. \hfill \Box

Motivated by the examples from the later parts of the previous section, we introduce the following concept:

**Definition 24.4** (Weak type-$(p,q)$) Given $p, q \in [1, \infty]$, an operator $T : \text{Dom}(T) \to L^0$ defined on a dense linear subspace $\text{Dom}(T) \subseteq L^p$, is said to be weak type-$(p,q)$ if
\[ \exists C \in (0, \infty) \forall f \in \text{Dom}(T) : \quad [Tf]_q \leq C \|f\|_p \tag{24.21} \]
Note that, for all \( p, q \in (0, \infty] \),
\[
T \text{ strong type } (p, q) \Rightarrow T \text{ weak type } (p, q).
\] (24.22)

Theorem 24.3 thus says that if a sublinear operator is weak type \((p_0, p_0)\) and \((p_1, p_1)\), then it is strong type \((p, p)\) for all \( p \in (p_0, p_1)\).

Note that the dependence of the coefficient on the right-hand side of (24.11) on those in (24.9–24.10) is the same as in (23.4) except for the \( p_0, p, p_1\)-dependent numerical prefactor. This prefactor diverges as \( p \downarrow p_0 \) or \( p \uparrow p_1 \) which is expected since convergence to a finite number as \( p \uparrow p_1 \) would imply (by a simple continuity argument) that \( T \) is strong type \((p_1, p_1)\), and this is more than we assume. On the other hand, when the operator is strong type at an endpoint, we expect the optimal prefactor to be bounded in the limit, very much how we argued right after the proof of Lemma 24.1.

As a direct application of Theorem 24.3, we get the proof of \( L^p\)-continuity of the map \( f \mapsto f^* \), where \( f^* \) is the Hardy-Littlewood maximal function:

**Proof of Theorem 7.11.** By Theorem 7.8, \( Tf := f^* \) is weak type \((1, 1)\) and, by inspection, it is strong type \((\infty, \infty)\) (which is the same as weak type \((\infty, \infty)\)). Then (24.11) gives
\[
\exists c \in (0, \infty) \forall p \in (1, \infty) \forall f \in L^p: \|f_p^*\| \leq \left(\frac{cp}{p-1}\right)^{1/p} \|f\|_p,
\] (24.23)
where \( c \) depends only on the dimension of the underlying Euclidean space. \( \square \)

### 24.2 Marcinkiewicz interpolation, general version.

While the diagonal version of Marcinkiewicz’ interpolation theorem appears quite regularly, at other times need arises for operators mapping \( L^{p_i} \to L^{q_i} \) for \( p_i \neq q_i \). The first important departure from Riesz-Thorin setting is that, should our main goal be to upgrade weak-type operators to strong-type ones, we cannot allow the indices to be so completely unrelated. Indeed, we have:

**Lemma 24.5** Let \( X = Y := \mathbb{N} \) be endowed with the \( \sigma \)-algebra \( 2^{\mathbb{N}} \) and, given any \( \beta > 0 \), consider the measures \( \mu \) and \( \nu \) defined by \( \mu(\{n\}) := 2^n \) and \( \nu(\{n\}) := 2^{\beta n} \). Let \( Tf := f \) be the identity map \( L^\beta(\mu) \to L^\beta(\nu) \). Then \( T \) is weak type \((p, \beta p)\) for each \( p > 0 \) yet, assuming \( \beta < 1 \), it is not strong type \((p, \beta p)\) for any \( p > 0 \).

**Proof.** For any \( f: \mathbb{N} \to \mathbb{R} \) and \( t > 0 \), \( \mu(\{|f| > t\}) \) is proportional to \( 2^{\max\{|n\in\mathbb{N}:|f(n)|>t\}} \) and similarly \( \nu(\{|f| > t\}) \) is proportional to \( 2^{\beta \max\{|n\in\mathbb{N}:|f(n)|>t\}} \). Hence, there is \( c \in (0, \infty) \) depending only on \( \beta \) such that
\[
\forall t > 0: \nu(\{|f| > t\})^{\frac{1}{\beta p}} \leq c \mu(\{|f| > t\})^{1/p}
\] (24.24)
This shows that \( |Tf|_{\beta p} \leq c |f|_{p} \leq c \|f\|_{p} \) and so \( T \) is weak type \((p, \beta p)\) for all \( p > 0 \). Next, for \( \beta < 1 \) let \( \alpha > 1 \) obey \( \alpha \beta < 1 \). Then \( f(n) := n^{-\alpha / p} 2^{-n/p} \) obeys \( \|f\|_{p} = \sum_{n \geq 1} n^{-\alpha} < \infty \) yet \( \|Tf\|_{\beta p} = \sum_{n \geq 1} n^{-\alpha \beta} = \infty \). Hence \( T \) is not strong type \((p, \beta p)\) for any \( p > 0 \). \( \square \)

This shows that we generally need to assume the inequalities \( p_0 < q_0 \) and \( p_1 < q_1 \). With this settled, we are ready to state:
The assumptions (24.27) give, for both

\[ p_0 \leq q_0 \land p_1 \leq q_1 \land q_0 \neq q_1. \]  

(24.25)

Let \( T : (L^{p_0} + L^{p_1})(X,F,\mu) \to L^0(Y,G,\nu) \) be sublinear and set, for \( \theta \in [0,1] \),

\[ \frac{1}{p_{\theta}} := \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q_{\theta}} := \frac{1 - \theta}{q_0} + \frac{\theta}{q_1} \]  

(24.26)

If \( T \) is weak type \( (p_0,q_0) \) and \( (p_1,q_1) \) then \( T \) is strong type \( (p_\theta,q_\theta) \) for all \( \theta \in (0,1) \). Explicitly, for all \( C_0,C_1 \in (0,\infty) \) and all \( \theta \in (0,1) \) there is \( C_\theta \in (0,\infty) \) such that

\[ \forall f \in L^{p_0} + L^{p_1}: \quad [Tf]_{q_\theta} \leq C_0 \|f\|_{p_0} \land [Tf]_{q_1} \leq C_1 \|f\|_{p_1} \]  

(24.27)

implies

\[ \forall f \in L^{p_0} + L^{p_1} \forall \theta \in (0,1): \quad \|Tf\|_{q_\theta} \leq C_\theta \|f\|_{p_\theta}. \]  

(24.28)

Proof for \( q_0,q_1 < \infty \). While a proof could still be based on the decomposition (24.12), the calculations become quite tedious due to the fact that the integral on the right of (24.14–24.15) now appears inside a power (namely \( q_0/p_0 \), resp., \( q_1/p_1 \)) which is larger than one. We will proceed by a different argument that relies on dyadic decompositions.

The case when \( p_0 = p_1 \) is handled directly by Lemma 24.1 so, by symmetry, we can assume \( p_0 \neq p_1 \) and \( q_0 < q_1 \) in what follows. Let \( f \) be a simple function satisfying \( \mu(\text{supp}(f)) < \infty \). By scaling and the homogeneity of \( T \) we may assume

\[ \|f\|_{p_\theta} \leq 1 \]  

(24.29)

with the aim of proving that \( \|Tf\|_{q_\theta} \) is then bounded by a universal constant. We will invoke a dyadic decomposition of the range of \( f \). Indeed, since \( f \) is finitely-valued, there is \( N \geq 1 \) such that

\[ f = \sum_{m=-N}^{N} f_m \quad \text{where} \quad f_m := f 1_{\{2^m < \|f\| \leq 2^{m+1}\}}. \]  

(24.30)

The subadditivity implies

\[ |Tf| \leq \sum_{m=-N}^{N} |Tf_m| \]  

(24.31)

and so, for any \( n \in \mathbb{Z} \) and any \( \{a_m\}_{m=-N}^{N} \) positive with \( \sum_{m=-N}^{N} a_m = 1 \),

\[ \nu(|Tf| > 2^n) \leq \sum_{m=-N}^{N} \nu(|Tf_m| > a_m 2^n) \]  

(24.32)

The assumptions (24.27) give, for both \( i = 0,1 \) and any \( t > 0 \),

\[ \nu(|Tf_m| > a_m t) \leq \left( \frac{C_i}{a_m t} \right)^{q_i} \|f_m\|_{p_i}^{q_i} \leq \left( \frac{C_i}{a_m t} \right)^{q_i} \|f_m\|_{p_i}^{q_i} \mu(\text{supp}(f_m))^{q_i/p_i} \leq \left( \frac{C_i}{a_m t} \right)^{q_i} 2^{(m+1)q_i} \mu(|f| > 2^m)^{q_i/p_i} \]  

(24.33)
Combining this with (24.32) we obtain
\[
\| T_f \|^q_{q_0} \leq \sum_{m=-N}^{N} \sum_{n \in \mathbb{Z}} 2^n \int_{t / a_m \leq 2^{n+1}} q_0 t^{q_0-1} \mu(|T f_m| > t) \, dt
\]
\[
\leq 2^{q_0} \sum_{m=-N}^{N} \sum_{n \in \mathbb{Z}} (2^n a_{n,m})^{q_0} \mu(|T f_m| > a_m 2^n)
\]
\[
\leq 2^{q_0} \sum_{n \in \mathbb{Z}} 2^{n q_0} \sum_{m=-N}^{N} a_m^{q_0} \min_{i=0,1} \left( (2 C_i)^{q_i} \left( \frac{2^{m-n}}{a_m} \right)^{q_i} \mu(|f| > 2^m) \frac{q_i}{p_i} \right)
\]
(24.34)

Now comes the time to use the conditions \( q_i \geq p_i \) to bound
\[
\mu(|f| > 2^m) \frac{q_i}{p_i} \leq \mu(|f| > 2^m) \left( 2^{-m p_0} |f| \right) \frac{q_0}{p_0} < 2^{q_0} \sum_{m=-N}^{N} 2^{m p_0} \mu(|f| > 2^m) R_m
\]
(24.36)

where
\[
R_m := a_m^{q_0} \sum_{n \in \mathbb{Z}} \min_{i=0,1} \left( (2 C_i)^{q_i} 2^{n(q_0-q_i)} 2^{m(1-p_0/p_1)} q_i \right)
\]
(24.37)

Our aim is to show that \( \{ R_m \}_{m=-N}^{N} \) is bounded regardless of \( \{ a_m \}_{m=-N}^{N}. \) For this we note that, by our assumptions, \( q_0 < q_0 < q_1. \) Pick any \( u > 0 \) and split the sum according to whether \( n \) obeys \( 2^n > u \) or not. Bounding the minimum by the \( i = 0 \) term in the former case and by the \( i = 1 \) term in the latter case yields
\[
R_m \leq \tilde{c} a_m^{q_0} \left( \left( \frac{2 C_0}{a_m} \right)^{q_0} u^{q_0-q_0} + \left( \frac{2 C_1}{a_m} \right)^{q_1} u^{q_1-q_1} \right)
\]
(24.38)

where
\[
\tilde{c} := \max_{i=0,1} \sum_{n \geq -1} 2^{-n(q_0-q_i)}.
\]
(24.39)

Here the sum starting from negative one to account for \( u \) not being of the form \( 2^n. \)

We now now minimize the quantity in (24.38) over \( u \) by noting that, for any \( \alpha, \gamma > 0 \) and \( A, B > 0, \)
\[
\inf_{u > 0} (Au^\alpha + Bu^{-\gamma}) = \Gamma(\alpha, \gamma) A \frac{\alpha}{\gamma+\alpha} B \frac{1}{\gamma+\alpha}
\]
(24.40)

where \( \Gamma(\alpha, \gamma) := (\alpha/\gamma)^\alpha + (\alpha/\gamma)^{-\gamma}. \) In our situation we have \( \alpha := q_0 - q_0 \) and \( \gamma := q_1 - q_0 \) and so
\[
R_m \leq \tilde{c} \Gamma(q_0 - q_0, q_1 - q_0) a_m^{q_0} \left( \frac{2 C_0}{a_m} 2^{m(1-p_0/p_0)} \right)^{q_0} \left( \frac{2 C_1}{a_m} 2^{m(1-p_0/p_1)} \right)^{q_1}
\]
(24.41)

A calculation shows that the first exponent on the right equals \( q_0 (1 - \theta) \) while the second equals \( q_0 \theta. \) This implies, somewhat unexpectedly, that all of the terms dependent on \( m \)
(including $a_m$) cancel out and we get
\[ R_m \leq \hat{c} \Gamma(q_\theta - q_0, q_1 - q_\theta) [2C_0^{1-\theta} C_1^{q_\theta}]^{q_\theta}. \] (24.42)
Using this in (24.36), we get that, for $f$ simple satisfying (24.29),
\[ \|Tf\|_{q_0} \leq 4 \hat{c} \Gamma(q_\theta - q_0, q_1 - q_\theta) [2C_0^{1-\theta} C_1^{q_\theta}]^{1/q_\theta} C_1^{q_\theta}. \] (24.43)
This implies the statement in (24.28).

We will leave the proof for $q_0 = q_1 = \infty$ to a homework assignment. The above proof does not even use the full power of weak-type property. Indeed, let:

**Definition 24.7** (Restricted weak type) We say that $T$ is restricted weak type $(p,q)$, if there exists $C \in (0,\infty)$ such that (for $q < \infty$)
\[ \forall t > 0: \quad \mu(|Tf| > t) \leq Ct^{-q}\|f\|^q_{\nu} \mu(\text{supp}(f))^{q/p}. \] (24.44)
If $q = \infty$ then we let the notion coincide with the weak/strong type $(p,q)$.

We then have:

**Corollary 24.8** The conclusion of Theorem 24.6 remains unchanged if the weak-type properties (24.27) are replaced by restricted weak-type properties (with $C$ in (24.44) given by $C_0^{q_0}$ and $C_1^{q_1}$, respectively).

**Proof.** The restricted weak-type properties imply directly the second line in (24.33) and thus bypass the use of the weak type properties (24.27).

The computation in (24.33) shows
\[ T \text{ weak type } (p,q) \Rightarrow T \text{ restricted weak type } (p,q) \] (24.45)
which further extends (24.22).

### 24.3 Some applications.

Let us now give some application of the Marcinkiewicz theorem. Recall that the Schur test for integral operators
\[ Tf(x) := \int K(x,y)f(y)\mu(dy) \] (24.46)
with kernel $K$ required that the $L^1$-norm in one variable is (essentially) bounded in the second variable and *vice versa*. With the help of interpolation, this can be extended to higher norms. Let us start with the “strong-type” form of the claim:

**Proposition 24.9** (Schur test extended) Let $(X,\mathcal{F},\mu)$ and $(Y,\mathcal{G},\nu)$ be $\sigma$-finite measure spaces and let $K: X \rightarrow Y \rightarrow \mathbb{R}$ be $\mathcal{F} \otimes \mathcal{G}$-measurable. Suppose that, for some $r,s \geq 1$,
\[ \exists C \in (0,\infty): \quad \|K(x,\cdot)\|_{L^r(\nu)} \leq C \quad \text{for } \mu\text{-a.e. } x \in X \] (24.47)
and
\[ \exists \tilde{C} \in (0,\infty): \quad \|K(\cdot,y)\|_{L^s(\mu)} \leq \tilde{C} \quad \text{for } \nu\text{-a.e. } y \in Y. \] (24.48)

Preliminary version (subject to change anytime!)
Then the operator in (24.46) is strong type \((p, q)\) for every \(p\) and \(q\) with
\[
1 \leq p \leq \frac{r}{r-1} \land s \leq q \leq \infty \land \frac{1}{p} + \frac{1}{r} = 1 + \frac{s}{r q}
\] (24.49)

Proof. First observe that, by Hölder’s inequality
\[
\|Tf\|_\infty \leq C\|f\|_{\frac{r}{r-1}}
\] (24.50)
and so \(T\) is strong type \((\frac{r}{r-1}, \infty)\). Moreover, Minkowski’s integral inequality (cf Lemma 22.21) implies
\[
\|Tf\|_s \leq \tilde{C}\|f\|_1
\] (24.51)
and so \(T\) is strong type \((1, s)\). Whenever there is \(\theta \in [0, 1]\) such that
\[
\frac{1}{p} = 1 - \theta + \theta \left(1 - \frac{1}{r}\right) \land \frac{1}{q} = \frac{1 - \theta}{s}
\] (24.52)
Riesz-Thorin interpolation theorem implies that \(T\) is strong type \((p, q)\). The condition (24.52) is now readily checked to be equivalent to (24.49).

Using Marcinkiewicz interpolation instead of Riesz-Thorin’s, we then get also a weak-type version of the Schur test:

**Proposition 24.10** (Schur test, weak-type version) Let \((X, \mathcal{F}, \mu)\) and \((Y, \mathcal{G}, \nu)\) be \(\sigma\)-finite measure spaces and let \(K: X \to Y \to \mathbb{R}\) be \(\mathcal{F} \otimes \mathcal{G}\)-measurable. Suppose that, for some \(r, s > 1,\)
\[
\exists C \in (0, \infty): \quad [K(x, \cdot)]_r \leq C \quad \text{for } \mu\text{-a.e. } x \in X
\] (24.53)
and
\[
\exists \tilde{C} \in (0, \infty): \quad [K(\cdot, y)]_s \leq \tilde{C} \quad \text{for } \nu\text{-a.e. } y \in Y.
\] (24.54)

Then the operator in (24.46) is strong type \((p, q)\) for every \(p\) and \(q\) satisfying
\[
1 < p < \frac{r}{r-1} \land s < q < \infty \land \frac{1}{p} + \frac{1}{r} = 1 + \frac{s}{r q}
\] (24.55)

Proof. We start by noting that, for any \(A \in \mathcal{G}\) with \(\nu(A) \in (0, \infty),\) Tonelli’s theorem shows along with the inequality (23.56) from Lemma 23.16 and (24.49) that
\[
\int_A \left( \int |K(x, y)| |f(y)| \mu(dy) \right) \nu(dx) \\
\leq \int \left( \int_A |K(x, y)| \nu(dx) \right) |f(y)| \mu(dy) \leq \frac{s}{s-1} \nu(A)^{1-1/s} \|f\|_1
\] (24.56)

Lemma 23.16 then gives \([Tf]_s \leq C \frac{s}{s-1} \|f\|_1\) and so \(T\) is weak type \((1, s)\).

For the other “boundary” weak-type property, we note that from
\[
\int |K(x, y)| |f(y)| \mu(dy) \leq \|f\|_\infty \int_{\text{supp}(f)} |K(x, y)| \mu(dy) \leq C \|f\|_\infty \mu(\text{supp}(f))^{1-1/r},
\] (24.57)
where we used (24.53) in the last step, we get that \(T\) is restricted weak type \((\frac{r}{r-1}, \infty)\). This is sufficient to get the statement of Marcinkiewicz interpolation theorem and so \(T\) is strong type \((p, q)\) for any \(p\) and \(q\) satisfying (24.55).
Note that, even though we allow more flexibility in the parameters, we still have $q > p$ unless $r = s = 1$ and so the case treated in Theorem 22.22 is the only one that permits iterations without additional assumptions on the underlying measure spaces. A particularly interesting case is when $X = Y := \mathbb{R}^d$ with $\mathcal{F} = \mathcal{G} := \mathcal{B}(\mathbb{R}^d)$ and $\mu = \nu := \lambda$ (with $\lambda$ denoting the Lebesgue measure) and

$$K(x, y) := \frac{1}{|x - y|^\alpha} \quad (24.58)$$

(For instance, for $d = 3$ and $\alpha = 1$, this corresponds to the electrostatic potential of a point charge.) Here we get:

**Corollary 24.11 (Hardy-Littlewood-Sobolev inequality)** For $\alpha \in (0, d)$ and $f \in L^1$ define

$$T_\alpha f(x) := \int_{\mathbb{R}^d} \frac{1}{|x - y|^{\alpha}} f(y) dy \quad (24.59)$$

Then $T_\alpha$ is strong type $(p, q)$ for $p \in (1, \frac{d}{d-\alpha})$ and $q \in (\frac{d}{\alpha}, \infty)$ such that $p^{-1} + \alpha/d = 1 + q^{-1}$. In particular, for each such $p, q$ there is $C \in (0, \infty)$ such that

$$\int f(x) \frac{1}{|x - y|^{\alpha}} g(y) \, dx \, dy \leq C \|f\|_p \|g\|_{\frac{q}{q-1}} \quad (24.60)$$

holds for all measurable $f, g : \mathbb{R}^d \to [0, \infty)$.

**Proof.** Since $\lambda(\frac{1}{|x|^\alpha} > t) = ct^{-d/\alpha}$, the bounds (24.53–24.54) hold with $r = s = d/\alpha$. The claim now follows from Proposition 24.10. The inequality (24.60) is a consequence of duality between $L^q$ and $L^{\frac{q}{q-1}}$.

Note that the strong form of Schur’s test cannot be used to prove the above corollary because $K(x, \cdot)$ is not in $L^p$ for any $p$. The inequality (24.60) is due to G.H. Hardy and J.E. Littlewood and, independently, S.L. Sobolev. The operator $T_\alpha$ is called the Riesz transform; the value $T_\alpha f$ is called the Riesz potential. These occur in connection with electrostatic theory. Indeed, when $f = g$ the quantity

$$\int f(x) \frac{1}{|x - y|^{\alpha}} f(y) \, dx \, dy \quad (24.61)$$

has the interpretation of a self-energy of an electrostatic potential $x \mapsto |x|^{-\alpha}$ under charge distribution $f(x) \, dx$. (Compare with our discussion of Riesz capacity in Definition 14.17.) In this case the finiteness of the integral for all $f \in L^p$ requires $p = \frac{d}{d-\alpha}$ which occurs when $p = \frac{2d}{2d-\alpha}$. In this case, E. Lieb determined the best constant $C$ explicitly and characterized the maximizing pairs $(f, g)$. These are not known for other values of $p$; see E. Lieb and M. Loss’ analysis textbook for full details.

Another operator of some interest to which the above theory can be applied is the so called fractional integral $I_\alpha f$ of $f$ which is defined, for $\alpha > 0$, by

$$I_\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} f(t) \, dt, \quad (24.62)$$
where $\Gamma$ is the Gamma function. It can be checked that (for $f$ bounded measurable)

$$\forall \alpha, \beta > 0: \quad I_\alpha(I_\beta f) = I_{\alpha + \beta} f.$$ (24.63)

Since $I_1 f = \int_0^\infty f(t)dt$, the map $a \mapsto I_\alpha f$ extends the sequence of iterated antiderivatives of $f$ to a continuous family. We then have:

**Lemma 24.12**  Regard $I_\alpha$ as an operator on subspaces of $L^0((0, \infty), B((0, \infty), 1_{(0, \infty)} \lambda))$. Let $\alpha \in (0, 1)$. Then $I_\alpha$ is weak type $(1, \frac{1}{1-\alpha})$ and strong type $(p, q)$ for all $p \in (1, \alpha^{-1})$ and $q$ defined by $q^{-1} = p^{-1} - \alpha$.

We leave the proof of this lemma to homework. Further applications of Marcinkiewicz interpolation will be discussed in the next section.