20. Duality and weak convergence

We will now move to discussing the spaces of continuous linear functionals on $L^p$; a topic that is central to the analysis in Banach spaces. We will then introduce the notion of weak convergence and give a number of important results that surround this topic.

20.1 Dual space to $L^p$.

The space of continuous functions $C(X)$ on any topological space is among the most natural structures to consider. Whenever $X$ is a vector space, it makes sense to restrict attention to linear functions. This leads to:

**Definition 20.1** (Continuous linear functionals) A functional $f: L^p \rightarrow \mathbb{R}$ is said to be

1. linear if it respects the linear structure of $L^p$, i.e.,
   $$ \forall f, g \in L^p \forall a, b \in \mathbb{R}: \phi(af + bg) = a\phi(f) + b\phi(g). $$

2. continuous if $\phi^{-1}(O)$ is open in $L^p$ for every open $O \subseteq \mathbb{R}$.

We will write $(L^p)^*$ for the space of continuous linear functionals on $L^p$. The space $(L^p)^*$ will sometimes be referred to as the dual of $L^p$.

To give an example, we note:

**Lemma 20.2** Consider a measure space $(X, \mathcal{F}, \mu)$ and let $p, q \in [1, \infty]$ be Hölder conjugate indices, i.e., $p^{-1} + q^{-1} = 1$. For any $g \in L^q$ set
   $$ \phi_g(f) := \int fg \, d\mu. $$

Then $\phi_g \in (L^p)^*$; i.e., $\phi_g$ is a continuous linear functional on $L^p$.

**Proof.** Hölder’s inequality shows
   $$ |\phi_g(f)| \leq \|f\|_p \|g\|_q $$

and, in particular, ensures that $\phi_g(f)$ is well-defined for all $f \in L^p$. Linearity is a consequence of linearity of the integral. For continuity, assume $g \neq 0$ (as $\phi_0 = 0$ which is continuous) and note that (20.3) shows that $\phi_g$ images a ball of radius $r$ around $f$ inside a ball of radius $r\|g\|_q$ around $\phi_g(f)$. If $O \subseteq \mathbb{R}$ contains $\phi_g(f)$, then it contains a ball of radius $\delta > 0$ around $\phi_g(f)$ and so its preimage contain a ball of radius $\delta/\|g\|_q$ around $f$. Hence $\phi_g$ is continuous as desired.

Note that the previous proof used implicitly the following concept:

**Definition 20.3** (Bounded linear functional) A linear functional $\phi: L^p \rightarrow \mathbb{R}$ is said to be bounded if there exists $c \in [0, \infty)$ such that
   $$ \forall f \in L^p: |\phi(f)| \leq c\|f\|_p. $$

We have already seen that being bounded implies being continuous. The converse is actually true as well, which we record as:

**Lemma 20.4** A linear functional $\phi: L^p \rightarrow \mathbb{R}$ is continuous if and only if it is bounded.
Proof. We have already shown in the proof of Lemma 20.2 that being bounded implies being continuous. For the converse, note that if \( \phi \) is continuous then \( \phi^{-1}((-1, 1)) \) is open and, since it contains 0, it contains a ball of radius \( r > 0 \) around 0. But then \( \|f\|_p < r \) implies \( |\phi(f)| \leq r \) and, by homogeneity, \( |\phi(f)| \leq r^{-1}\|f\|_p \) holds for all \( f \in L^p \).

The next question is we wish to address is how exactly is the best constant \( c \) in (20.4) related to \( \phi \). The fact that we work with linear functions on a linear space gives:

**Proposition 20.5** The space \( (L^p)^* \) is a linear vector space with addition and scalar multiplication defined by \((\phi + \psi)(f) := \phi(f) + \psi(f)\) and \((\alpha\phi)(f) := \alpha\phi(f)\). Moreover, denoting

\[
\|\phi\| := \sup_{f \in L^p \setminus \{0\}} \frac{|\phi(f)|}{\|f\|_p}
\]

(20.5)
defines a norm on \( (L^p)^* \). The space \( (L^p)^* \) is complete in this norm.

Proof. Linearity is checked readily so all we have to show is that \( \phi \mapsto \|\phi\| \) is a norm. Here positivity and triangle inequality are inferred from the properties of absolute value while \( \|\phi\| = 0 \) implies \( \phi = 0 \) by the very definition (20.5).

Suppose now \( \{\phi_n\}_{n \geq 1} \subseteq (L^p)^* \) is a Cauchy sequence in the norm \( \|\cdot\| \). The linearity implies

\[
\forall f \in L^p : \|\phi_n(f) - \phi_m(f)\| = \|\phi_n - \phi_m\|(f) \leq \|\phi_n - \phi_m\|\|f\|_p \tag{20.6}
\]

and so, for each \( f \in L^p \), the sequence \( \{\phi_n(f)\}_{n \geq 1} \) is Cauchy in \( \mathbb{R} \) and thus convergent. Define

\[
\phi(f) := \lim_{n \to \infty} \phi_n(f) \tag{20.7}
\]

Then \( \phi \) is linear. Since the triangle inequality for \( \|\cdot\| \) implies that \( c := \lim_{n \to \infty} \|\phi_n\| \) exists, we get \( |\phi(f)| \leq c\|f\|_p \) for each \( f \in L^p \). Hence, \( \phi \) is bounded and (by Lemma 20.4) continuous, implying \( \phi \in (L^p)^* \).

It remains to prove that \( \|\phi_n - \phi\| \to 0 \) as \( n \to \infty \). For this note that, for all \( n \geq 1 \),

\[
\forall f \in L^p : \|\phi(f) - \phi_n(f)\| \leq \lim_{m \to \infty} \|\phi_m(f) - \phi_n(f)\| \leq \lim_{m \to \infty} \|\phi_m - \phi_n\|\|f\|_p. \tag{20.8}
\]

So

\[
\|\phi - \phi_n\| \leq \lim_{m \to \infty} \|\phi_m - \phi_n\|. \tag{20.9}
\]

Taking \( n \to \infty \), the right-hand side tends to zero by the assumed Cauchy property of \( \{\phi_n\}_{n \geq 1} \). Hence \( \phi_n \to \phi \) in norm \( \|\cdot\| \). \( \square \)

Notice that the previous proof used nothing specific about \( L^p \)-spaces. Indeed, the conclusion of Proposition 20.5 applies in full generality of normed linear spaces and that even to those that are not complete.

A question that we wish to address is whether all continuous linear functionals on \( L^p \) take the form (20.2). The affirmative part of the answer is the content of:

**Theorem 20.6** (Riesz representation theorem) Consider the \( L^p \) spaces over a measure space \((X, \mathcal{F}, \mu)\). For \( p \in (1, \infty) \), the map \( g \mapsto \phi_g \) is a linear bijection \( L^q \to (L^p)^* \). Moreover, the map is an isometry,

\[
\forall g \in L^q : \|\phi_g\| = \|g\|_q. \tag{20.10}
\]
Assuming that \( m \) is \( \sigma \)-finite, the same holds also for \( p = 1 \).

The proof will need:

**Lemma 20.7 (Differentiability of \( L^p \)-norms)** For each \( p \in (1, \infty) \) and each \( f, h \in L^p \),

\[
\frac{d}{dt} \| f + th \|_p^p \bigg|_{t=0} = p \int |f|^{p-2}fh \, d\mu
\]  \tag{20.11}

**Proof.** The function \( F(t) := |f + th|^p \) is continuously differentiable with \( F'(t) = pf + th|^{p-2}[f + th]h \). The convexity of \( x \mapsto x^p \) in turn shows that the derivative is non-decreasing in \( t \) and \( F'(0) \leq \frac{1}{p}[F(t) - F(0)] \leq F'(t) \) for any \( t \geq 0 \). Integrating over the arguments of \( f \) and \( g \) gives

\[
p \int |f|^{p-2}fh \, d\mu \leq \frac{\| f + th \|_p^p - \| f \|_p^p}{t} \leq p \int |f + th|^{p-2}[f + th]h \, d\mu
\]  \tag{20.12}

Applying Dominated Convergence Theorem for the limit \( t \downarrow 0 \), we get (20.11). (The limit \( t \uparrow 0 \) is handled by reversing the sign of \( h \).) \( \square \)

**Proof of Theorem 20.6 for \( 1 < p < \infty \).** Let \( p \in (1, \infty) \) and \( \phi \in (L^p)^* \). We start by some general observations. Let

\[
K := \{ h \in L^p : \phi(h) = 0 \}
\]  \tag{20.13}

be the null space of \( \phi \). This is a closed linear subspace of \( L^p \) which, if \( \phi \neq 0 \), is of “codimension-1” meaning that, given any \( f \notin K \), each \( f' \in L^p \) can be written as a linear combination of \( f \) and an element from \( K \); specifically,

\[
f' = \frac{\phi(f')}{\phi(f)} f + h \quad \text{where} \quad h \in K.
\]  \tag{20.14}

Turning this around, if \( f' = af + h \) then \( \phi(f') = a\phi(f) \) and so \( \phi \) is determined its value at one single \( f \notin K \). The key point is to choose this \( f \) optimally which is where the uniform convexity of \( L^p \) will come handy.

Assume \( \phi \neq 0 \) and pick any \( f_0 \in L^p \setminus K \). Then \( \phi(f_0) \neq 0 \); by adjusting the sign of \( f_0 \) we may assume \( \phi(f_0) > 0 \). As \( K \) is closed and convex, by Proposition 19.15 there exists \( h_0 \in K \) such that

\[
\| f_0 - h_0 \|_p = \inf_{h \in K} \| f_0 - h \|_p.
\]  \tag{20.15}

Denote \( f := f_0 - h_0 \) and observe that \( \phi(f) = \phi(f_0) > 0 \). Since \( h \mapsto \| f - h \|_p \) is minimized at \( h = 0 \), Lemma 20.7 also implies

\[
\forall h \in K : \int |f|^{p-2}fh \, d\mu = 0.
\]  \tag{20.16}

We claim that

\[
\phi(f) = \| \phi \| \| f \|_p.
\]  \tag{20.17}

Indeed, if \( \phi(f) < \| \phi \| \| f \|_p \) were true, then (20.5) along with a simple scaling argument would yield the existence of \( f' \in L^p \) with \( \phi(f') = \phi(f) \) and \( \| f' \|_p < \| f \|_p \). By (20.14) we would then have \( f' = f_0 - h \) for some \( h \in K \), in contradiction with (20.15).
Noting that (20.17) determines the value of \( \phi(f) \), we now set
\[
g(x) := \frac{\|\phi\|_{p} f(x)}{\|f\|_{p}^{p-1}} f(x)|f(x)|^{p-2} \tag{20.18}
\]
A calculation shows that \( \phi_{g}(f) = \|\phi\|_{p} f \) while (20.16) implies that \( \phi_{g}(h) = 0 \) for all \( h \in K \). Hence \( \phi = \phi_{g} \). This shows that the map \( g \mapsto \phi_{g} \) is surjective. To see that it is injective we note that \( q(p-1) = p \) and so
\[
\|g\|_{q} = \frac{\|\phi\|_{p}}{\|f\|_{p}^{p-1}} \left( \int |f|^{q(p-1)} d\mu \right)^{1/q} = \|\phi\|. \tag{20.19}
\]
This shows that the map is injective. Linearity is verified directly from (20.2).

Note that the above proof covers the proof of Lemma 1.12, dealing with the special case of \( p = 2 \). As \( L^1 \) is not uniformly convex and, in fact, the conclusion of Proposition 19.15 fails in general for \( L^1 \), we have to supply a separate argument for \( p = 1 \).

**Proof of Theorem 20.6 for \( p = 1 \).** Suppose first that \( \mu \) is finite and let \( \phi \in (L^1)^{*} \). Then \( \|f\|_1 \leq \mu(X)^{1/2}\|f\|_2 \) for all \( f \in L^2 \) and, since \( L^2 \subseteq L^1 \) here (cf Lemma 19.8(1)), we have
\[
\forall f \in L^2 : \quad |\phi(f)| \leq \|\phi\| \mu(X)^{1/2}\|f\|_2. \tag{20.20}
\]
In follows that the restriction of \( \phi \) to \( L^2 \) is bounded, and thus continuous by Lemma 20.4. By the first part of the theorem, there exists \( g \in L^2 \) such that
\[
\forall f \in L^2 : \quad \phi(f) = \int fg \, d\mu. \tag{20.21}
\]
We now claim that \( g \in L^\infty \). Taking \( f := 1_A \) with \( A := \{g > \|\phi\|+\delta\} \) gives \( \phi(1_A) \geq (\|\phi\|+\delta)\mu(A) \) which in light of \( \phi(1_A) \leq \|\phi\|\mu(A) = 0 \). Employing a similar argument for \( -g \) instead of \( g \) then gives \( \|g\|_{\infty} \leq \|\phi\| \).

We now use these facts to show that (20.21) applies to all \( f \in L^1 \) and that \( \|g\|_{\infty} = \|\phi\| \). For the first part we note that any \( f \in L^1 \) is an \( L^1 \)-limit of \( f_n := f 1_{\{|f| \leq n\}} \). By continuity of \( \phi \) we have \( \phi(f) = \lim_{n \to \infty} \phi(f_n) \) and, in light of a.e.-boundedness of \( g \), we have \( \{ f_n g \, d\mu \to \{ f g \, d\mu \). Hence (20.21) holds for all \( f \in L^1 \) and so \( \phi = \phi_{g} \) and the map \( g \mapsto \phi_{g} \) is surjective. Hölder’s inequality then shows \( |\phi(f)| \leq \|g\|_{\infty}\|f\|_1 \) and so \( \|\phi\| \leq \|g\|_{\infty} \) and the map is injective.

This completes the proof for the finite measure case. In the \( \sigma \)-finite case, we split \( X \) into a countable family of disjoint sets of finite measure and perform the same argument on each piece separately. This readily extends surjectivity and, in light of norm preservation, also injectivity.

Before we move to the situations when the above question is answered negatively, let us pause to note that another line of argument exists that proves Theorem 20.6 based on the Radon-Nikodým Theorem. This works best for \( p = 1 \) and finite underlying measures. Indeed, \( \phi \in (L^1)^{*} \) then defines a finite measure via
\[
\nu(A) := \phi(1_A) \tag{20.22}
\]
and, since \(|\phi(1_A)| \leq \|\phi\| \mu(A)\), the Radon-Nikodym theorem gives

\[ \phi(1_A) = \int_A g \, d\mu \]  

(20.23)

with \(g \in L^\infty\). Using \(L^1\) convergence, we then extend (20.23) to the desired from (20.2). Extensions to all \(L^p\) cases then proceed via the argument used in (20.20). Some work is required to remove the finite-measure assumption when \(p > 1\). As we show below, the \(L^1\)-case fails for general measure spaces.

Somewhat circuitously, our first proof of the Radon-Nikodym theorem in 245B was actually based on \(p = 2\) version of Theorem 20.6. (We then supplied another proof, which relied entirely on signed measures.) The \(p = 2\) part of Theorem 20.6 can be proved directly, even without the need of Proposition 19.15. In any case, the Riesz representation theorem and the Radon-Nikodym theorem are very closely related.

Let us now move to various counterexamples, all drawn from Folland’s book:

**Lemma 20.8** Suppose \(\mu\) is a measure on \((X, \mathcal{F})\) such that some \(A \in \mathcal{F}\) contains no non-empty measurable subset of finite measure. Then \(\phi_{1_A} = 0\) yet \(1_A \in L^\infty\) is non-zero. Thus \(g \mapsto \phi_g\) as a map \(L^\infty \to (L^1)^*\) is not injective and, in particular, not isometric.

**Proof.** If \(L^1 = \{0\}\) then \((L^1)^* = \{0\}\), so let us assume \(L^1 \neq \emptyset\). Pick \(f \in L^1\). Then \(|\{f > \epsilon\}| = \) of finite measure for all \(\epsilon > 0\). Hence, by taking the union over \(\epsilon\) tending to zero we get that \(\mu(A \cap \{f \neq 0\}) = 0\) for each \(f \in L^1\). This gives \(\phi_{1_A} = 0\), yet \(A\) is of infinite measure and so \(1_A \neq 0\) in \(L^\infty\). \(\square\)

In Folland’s book, the measures avoiding pathological sets discussed in the previous lemma are given a name:

**Definition 20.9** We say that a measure \(\mu\) on \((X, \mathcal{F})\) is semifinite if for all \(A \in \mathcal{F}\),

\[ \mu(A) = \infty \Rightarrow \exists B \in \mathcal{F}: B \subseteq A \land 0 < \mu(B) < \infty. \]  

(20.24)

Folland also shows that, with a suitably modified definition of the \(L^\infty\)-space, the above example can be avoided in general. Concerning surjectivity of the embedding of \(L^\infty\) into \((L^1)^*\), we get:

**Lemma 20.10** Let \(X\) be an uncountable set, \(\mathcal{F} = 2^X\) and \(\mu\) the counting measure. Let \(\mathcal{F}_0\) be the \(\sigma\)-algebra of countable and co-countable sets and \(\mu_0\) the counting measure on \(\mathcal{F}_0\). Then

\[ L^1(\mu) = L^1(\mu_0) \land L^1(\mu)^* = L^1(\mu_0)^* = L^\infty(\mu) \neq L^\infty(\mu_0) \]  

(20.25)

In particular, \(g \mapsto \phi_g\) is not surjective as the map \(L^\infty(\mu_0) \to L^1(\mu_0)^*\).

**Proof.** Since the counting measure vanishes only on the empty set, the \(L^p\)-spaces do not require equivalence classes. If \(f\) is integrable with respect to the counting measure, then \(\{f \neq 0\}\) is countable and so it belongs to \(\mathcal{F}_0\). Hence, \(f\) is \(\mathcal{F}_0\)-measurable. Thus \(L^1(\mu_0) = L^1(\mu)\). On the other hand, since the counting measure vanishes only on the empty set, \(L^\infty(\mu)\) contains all bounded functions while \(L^\infty(\mu_0)\) contains only those bounded functions that are \(\mathcal{F}_0\) measurable, which are those constant on a co-countable set. As \(X\) is uncountable, \(L^\infty(\mu_0)\) is a proper subset of \(L^\infty(\mu)\). \(\square\)
Some of the conclusions of Theorem 20.6 extend even to \( p = \infty \). Indeed, the map 
\( g \mapsto \phi_s \) is always an isometry \( L^1 \to (L^\infty)^\ast \) and, in particular, it is injective. However, surjectivity fails in general and provably (assuming suitable axioms in the set theory) whenever \( X \) partitions into infinitely many sets of positive measure. We will return to this after we have proved the Hahn-Banach theorem.

### 20.2 Weak convergence and topology.

We will now return to the original idea of studying a set \( X \) (equipped with an “original” topology) using the associated space \( C(X) \) of continuous functions. Note that every function \( f \in C(X) \) induces a natural topology on \( X \); namely, the coarsest topology \( \{ f^{-1}(O) : O \subseteq \mathbb{R} \text{ open} \} \) that makes \( f \) continuous. Forcing these to be included for all \( f \in C(X) \) tends to reproduce the original topology; for instance, if \( X \) is a metric space and \( O \subseteq X \) is open, then \( f(x) := \text{dist}(x, O^c) \) is continuous with \( f^{-1}((0, \infty)) = O \). However, interesting topologies do arise when we restrict attention to subclasses of \( C(X) \); notably, linear functions if \( X \) is a vector space. This leads to:

**Definition 20.11 (Weak topology)** Let \( V \) be a normed space over \( \mathbb{R} \) and let \( V^\ast \) be the space of continuous linear functionals on \( V \). The coarsest topology containing

\[
\{ \phi^{-1}(O) : O \subseteq \mathbb{R} \wedge \phi \in V^\ast \}
\]

is called the weak topology on \( V \).

While the above definition is specific to vector spaces, it mimics closely the general definition used in topology. (A standard use of weak topology there is in the definition of the product topology, which is the weak topology induced by the family of coordinate projections.) We caution the reader that the weak topology is generally not first countable (and thus not metrizable), which means that convergence must be formulated using nets, rather than just sequences. Notwithstanding, we put forward:

**Definition 20.12 (Weakly convergent sequences)** A sequence \( \{x_n\}_{n \geq 1} \subseteq V \) is said to be weakly convergent to \( x \in V \) (with notation \( x_n \xrightarrow{w} x \)) if

\[
\forall \phi \in V^\ast : \quad \phi(x) = \lim_{n \to \infty} \phi(x_n)
\]

It is clear that (strongly) convergent sequences are weakly convergent. To appreciate the full depth of the concept, let \( f_n : [0, 1] \to \mathbb{R} \) be defined by

\[
f_n(x) := \begin{cases} 
1, & \text{if } [2^n x] \text{ is even,} \\
0, & \text{else.}
\end{cases}
\]

Writing \( \lambda \) is the Lebesgue measure, then \( f_n \in L^p([0, 1], \lambda) \) for all \( p \in [1, \infty] \). For any \( g : [0, 1] \to \mathbb{R} \) continuous, elementary approximation arguments show

\[
\int_{[0,1]} g f_n \, d\lambda \xrightarrow{n \to \infty} \frac{1}{2} \int_{[0,1]} g \, d\lambda
\]
and, as any $L^q$-function (for all $q \in [1, \infty]$) can be approximated by a continuous function inside the integrals, this extends to all $g \in L^q$. We conclude that

$$\forall p \in [1, \infty): \quad f_n \rightharpoonup \frac{1}{2} 1_{[0,1]} \quad \text{in } L^p$$  \hspace{1cm} (20.30)

As this $\{f_n\}_{n \geq 1}$ converges neither pointwise, nor a.e., nor in measure and not even in $L^p$, we see that weak convergence and topology is thus weaker than any of the notions of convergence we have used before. (We note that (20.30) fails for $p = \infty$ because, by the Hahn-Banach theorem, $(L^\infty)^*$ contains $\phi$ such that $\phi(f_{2n}) = 1$ and $\phi(f_{2n+1}) = 0$.)

A careful reader will notice a caveat in the above definition and example: is the weak limit even unique? As it turns out, this is related to the question whether the weak topology separates points meaning that it is Hausdorff (or $T_2$). Keeping our main theme, we will work this out for the specific example of $L^p$-spaces:

**Lemma 20.13** (Continuous linear functionals on $L^p$ separate points) Let $p \in [1, \infty)$. Then

$$\forall f \in L^p \setminus \{0\} \exists \phi \in (L^p)^*: \quad \phi(f) \neq 0. \quad \text{(20.31)}$$

If $\mu$ is semifinite (see Definition 20.9), then the same holds for $p = \infty$.

**Proof.** Consider first $p \in [1, \infty)$ and let $f \in L^p \setminus \{0\}$. Setting $g := |f|^{p-2}f$, we get $g \in L^q$ and so $\phi_g \in (L^p)^*$ by Lemma 20.2. A calculation shows $\phi_g(f) = \|f\|^p \neq 0$. For $p = \infty$, if $f \in L^\infty$ is non-zero, then semifiniteness of $\mu$ implies existence of $\epsilon > 0$ and a measurable $A \subseteq \{|f| > \epsilon\}$ with $\mu(A) \in (0, \infty)$. Taking $g := \text{sign}(f) 1_A$ we get $\phi_g(f) > 0$. \hfill $\Box$

The reader should not read the $p = \infty$ part as being false when $\mu$ is not semifinite, although the argument in the proof of Lemma 20.8 may seem suggestive of that. Indeed, the Hahn-Banach theorem permits us to construct $\phi \in (L^\infty)^*$ such that $\phi(1_A) > 0$ even if $A$ does not contain any sets of finite and positive measure. Lemma 20.8 just says that this $\phi$ cannot be of the form $\phi_g$ with $g \in L^1$.

Moving along our exploration of weak convergence, our second point of concern is whether weakly convergent sequences are bounded. For this we introduce:

**Definition 20.14** (Weak boundedness) A set $A \subseteq L^p$ is weakly bounded if

$$\phi \in (L^p)^*: \quad \sup_{f \in A} |\phi(f)| < \infty. \quad \text{(20.32)}$$

Since

$$\sup_{f \in A} |\phi(f)| \leq \|\phi\| \sup_{f \in A} \|f\|_p \quad \text{(20.33)}$$

if $A \subseteq L^p$ is bounded (in norm topology on $L^p$), then it is weakly bounded. The converse is true as well, but not without work:

**Theorem 20.15** (Uniform boundedness principle in $L^p$) Let $p \in [1, \infty)$. Then for all non-empty $A \subseteq L^p$,

$$\left( \forall \phi \in (L^p)^*: \quad \sup_{f \in A} |\phi(f)| < \infty \right) \quad \Rightarrow \quad \sup_{f \in A} \|f\| < \infty. \quad \text{(20.34)}$$
In particular, every weakly bounded subset of $L^p$ is bounded in $L^p$. The same holds for $p = \infty$ whenever $\mu$ is semifinite.

**Proof.** Let us again start first with $p \in [1, \infty)$. We will aim to prove the contrapositive, so suppose $\sup_{f \in A} \|f\| = \infty$. Then there is $\{f_n\}_{n \geq 1} \subseteq A$ such that

$$\lim_{n \to \infty} 3^{-n} \|f_n\|_p = \infty. \quad (20.35)$$

Assuming also that $\|f_n\| \neq 0$ for all $n \geq 1$, consider the functions

$$g_n := \frac{1}{\|f_n\|_p^{p-1}} |f_n|^{p-2} f_n. \quad (20.36)$$

where for $p = 1$ we interpret the right-hand side as $\text{sign}(f)$, and note that $g_n \in L^q$ and so, by Lemma 20.2, $\phi_{g_n} \in (L^p)^*$ with the explicit value of the norms,

$$\|\phi_{g_n}\| = \|g_n\|_q = 1. \quad (20.37)$$

Next we define a sequence $\{\sigma_n\}_{n \geq 1} \subseteq \{-1, +1\}^N$ recursively by $\sigma_1 := +1$ and

$$\forall n \geq 2: \quad \sigma_n \phi_{g_n} (f_n) \left( \sum_{k=1}^{n-1} 3^{-k} \sigma_k \phi_k (f_n) \right) \geq 0. \quad (20.38)$$

Define

$$\phi := \sum_{n \geq 1} 3^{-n} \sigma_n \phi_{g_n}. \quad (20.39)$$

The sum converges in $\|\cdot\|_p$-norm on $(L^p)^*$ by (20.37) and so $\phi \in (L^p)^*$ by the completeness of $(L^p)^*$. We now observe

$$|\phi(f_n)| \geq 3^{-n} \sigma_n \phi_{g_n} (f_n) + \sum_{k=1}^{n-1} 3^{-k} \sigma_k \phi_{g_k} (f_n) - \sum_{k>n} 3^{-k} \sigma_k \phi_{g_k} (f_n)$$

$$\geq 3^{-n} \phi_{g_n} (f_n) - \sum_{k>n} 3^{-k} \|\phi_{g_k}\| \|f_n\|_p$$

$$= 3^{-n} \left( \|\phi_{g_n} (f_n)\| - \frac{1}{2} \|f_n\|_p \right), \quad (20.40)$$

where we used (20.38) to effectively disregard the second term in the first large absolute value and then applied the triangle inequality along with (20.37). From

$$\phi_{g_n} (f_n) = \|f_n\|_p \quad (20.41)$$

we conclude that $|\phi(f_n)| \geq \frac{1}{2} 3^{-n} \|f_n\|_p$. Then (20.35) shows $\phi(f_n) \to \infty$ and so we have $\sup_{f \in A} |\phi(f)| = \infty$, as desired.

For $p = \infty$ we start as before by finding $\{f_n\}_{n \geq 1} \subseteq L^\infty$ with $3^{-n} \|f_n\|_\infty \to \infty$. For each $n \geq 1$, the semifiniteness of $\mu$ ensures existence of a measurable set

$$A_n \subseteq \{ |f_n| > \frac{2}{3} \|f_n\|_\infty \} \quad (20.42)$$

with $\mu(A_n) \in (0, \infty)$. Then

$$g_n := \frac{1}{\mu(A_n)} 1_{A_n} \in L^1 \quad (20.43)$$

Preliminary version (subject to change anytime!) Typeset: April 29, 2020
obeys $\|\phi_{g_n}\| = \|g_n\|_1 = 1$. The rest of the proof proceeds along the same lines except that now $\phi_{g_n}(f_n) \geq \frac{2}{3} \|f_n\|_\infty$ and so $|\phi(f_n)| \geq \frac{1}{6} 3^{-n} \|f_n\|_\infty$. 

The above proof of the Uniform Boundedness Principle is specific to $L^p$ spaces but the argument applies in full generality of Banach spaces. The above technique is an example of a “gliding hump” (or “sliding hump”) argument; another line of proof relies on the Baire Category Theorem. The “sliding hump” argument also underlies the proof of:

**Corollary 20.16 (Principle of condensation of singularities)** Assume either $p \in [1, \infty)$ or $p = \infty$ with the underlying measure semifinite and let $\{\phi_{i,j}: i, j \geq 1\} \subset (L^p)^*$ be such that

$$\forall i \geq 1 \exists f_i \in L^p: \quad \sup_{j \geq 1} |\phi_{i,j}(f_i)| = \infty$$

Then

$$\exists f \in L^p \quad \forall i \geq 1: \quad \sup_{j \geq 1} |\phi_{i,j}(f_i)| = \infty.$$  

We leave the details of the proof to homework. The Uniform Boundedness Principle implies that all weakly convergent sequences are norm bounded. Another application is the content of:

**Corollary 20.17** Assume $1 < p < \infty$ and let $\{\phi_n\}_{n \geq 1} \subset (L^p)^*$ be such that

$$\forall f \in L^p: \quad \phi(f) := \lim_{n \to \infty} \phi_n(f) \text{ exists in } \mathbb{R}.$$  

Then $\phi \in (L^p)^*$. 

**Proof.** The limit functional $f \mapsto \phi(f)$ is clearly linear so that main point is to prove continuity. For $1 < p < \infty$, the Riesz representation theorem states that the double dual space $(L^p)^{**}$, which is the space of continuous linear functionals on $(L^p)^*$, is isometric to $L^q$. For each $f \in L^p$, the evaluation map $f \mapsto \phi(f)$ then defines a continuous linear functional on $(L^p)^*$. The statement (20.46) implies that $\{\phi_n\}_{n \geq 1}$ is weakly bounded in $(L^p)^*$, which is isometric to $L^q$ for $q$ the Hölder conjugate of $p$. The Uniform Boundedness Principle for $L^q$ gives $c := \sup_{n \geq 1} |\phi_n| < \infty$. As (20.46) gives $|\phi(f)| \leq c \|f\|_p$, this implies boundedness and, by Lemma 20.4, also continuity of $\phi$. 

The above argument used $L^p$-spaces with $1 < p < \infty$ only to the point that they adhere to the following concept:

**Definition 20.18 (Reflexive spaces)** A normed vector space $V$ is said to be reflexive if the evaluation map $x \mapsto x^{**}$ defined by

$$\forall \phi \in V^*: \quad x^{**}(\phi) := \phi(x)$$

images $V$ onto the double dual $V^{**}$. 

We note that the evaluation map is injective and continuous because $|x^{**}(\phi)| \leq \|x\| \|\phi\|$ (which shows $\|x^{**}\| \leq \|x\|$). In fact, it is an isometry because $\|x^{**}\| = \|x\|$ which we will prove in full generality later. A key point is that $L^p$ spaces are reflexive for $1 < p < \infty$ but generally are not for $p = 1$ and $p = \infty$. 

Preliminary version (subject to change anytime!)
The Uniform Boundedness Principle plays a fundamental role in the theory of linear operators on Banach spaces. The “pure analysis” applications include $L^p$-convergence of Fourier series and analytic families of linear operators.