Duality and weak convergence

Updated April 29, 2020
Outline:

- Continuous linear functionals over $L^p$
- Duality of $L^p$ with $L^q$ with $p^{-1} + q^{-1} = 1$
- Weak topology and convergence
- Uniform boundedness principle
- Reflexivity of $L^p$ for $1 < p < \infty$
Natural idea: if $X$ topological space, study $C(X)$
If $X =$ vector space, then consider linear functions

Definition (Continuous linear functionals)
A functional $\phi : L^p \to \mathbb{R}$ is said to be
(1) linear if it respects the linear structure of $L^p$, i.e.,
\[ \forall f, g \in L^p \ \forall a, b \in \mathbb{R} : \ \phi(af + bg) = a\phi(f) + b\phi(g). \]
(2) continuous if $\phi^{-1}(O)$ is open in $L^p$ for every open $O \subseteq \mathbb{R}$. 
Notation: $(L^p)^* :=$ set of continuous linear functionals on $L^p$
Lemma

Consider a measure space \((X, \mathcal{F}, \mu)\) and let \(p, q \in [1, \infty]\) be Hölder conjugate indices, i.e., \(p^{-1} + q^{-1} = 1\). For any \(g \in L^q\) set

\[
\phi_g(f) := \int fg \, d\mu
\]

Then \(\phi_g \in (L^p)^*\); i.e., \(\phi_g\) is a continuous linear functional on \(L^p\).

Proof: Hölder gives

\[
|\phi_g(f)| \leq \|f\|_p \|g\|_q
\]

so integral well defined for all \(f \in L^p\). Linearity clear. For continuity, note that \(\phi_g\) images \(B_X(f, r)\) into \(B_{\mathbb{R}}(\phi_g(f), r\|g\|_q)\). \(\square\)
Continuity and boundedness

Definition (Bounded linear functional)
A linear functional \( \phi : L^p \rightarrow \mathbb{R} \) is bounded if

\[
\exists c \in [0, \infty) \; \forall f \in L^p : \quad |\phi(f)| \leq c \|f\|_p.
\]

We then observe:

Lemma

For any linear functional \( \phi : L^p \rightarrow \mathbb{R} : \)

\( \phi \) is continuous \( \iff \phi \) is bounded

Proof: \( \Leftarrow \) proved above. For \( \Rightarrow \) note that \( \phi^{-1}((-1, 1)) \) contains 0 and thus \( B_X(0, r) \) for some \( r > 0 \). Then \( \|f\|_p < r \) implies \( |\phi(f)| \leq 1 \) and, by homogeneity, \( |\phi(f)| \leq r^{-1} \|f\|_p. \) \( \square \)
$(L^p)^*$ as a complete normed space

**Proposition**

$(L^p)^*$ is a linear vector space with addition and scalar multiplication defined by $(\phi + \psi)(f) := \phi(f) + \psi(f)$ and $(a\phi)(f) := a\phi(f)$. Moreover, denoting

$$
\|\phi\| := \sup_{f \in L^p \setminus \{0\}} \frac{|\phi(f)|}{\|f\|_p}
$$

defines a norm on $(L^p)^*$. The space $(L^p)^*$ is complete in this norm.
Proof of Proposition

Linearity and properties of norm checked readily, so main task is to show completeness.

Suppose \( \{\phi_n\}_{n \geq 1} \subseteq (L^p)^* \) Cauchy in norm \( \| \cdot \| \). Linearity implies

\[
\forall f \in L^p: \quad |\phi_n(f) - \phi_m(f)| = |(\phi_n - \phi_m)(f)| \leq \|\phi_n - \phi_m\| \|f\|_p
\]

so, for each \( f \in L^p \), the sequence \( \{\phi_n(f)\}_{n \geq 1} \) is Cauchy in \( \mathbb{R} \). Set

\[
\phi(f) := \lim_{n \to \infty} \phi_n(f)
\]

Then \( \phi \) linear and obeys

\[
|\phi(f)| \leq c \|f\|_p
\]

with \( c := \lim_{n \to \infty} \|\phi_n\| \). So \( \phi \) bounded and so continuous. It remains to show \( \|\phi_n - \phi\| \to 0 \) as \( n \to \infty \) …
...For this note that, for all $n \geq 1$ and all $f \in L^p$, 

$$|\phi(f) - \phi_n(f)| = \lim_{m \to \infty} |\phi_m(f) - \phi_n(f)| \leq \lim_{m \to \infty} \|\phi_m - \phi_n\| \|f\|_p$$

So 

$$\|\phi - \phi_n\| \leq \lim_{m \to \infty} \|\phi_m - \phi_n\|$$

Taking $n \to \infty$, the RHS tends to zero by the assumed Cauchy property of $\{\phi_n\}_{n \geq 1}$. Hence $\phi_n \to \phi$ in norm $\| \cdot \|$ $\square$

Note: noting specific to $L^p$ above, works for all normed spaces!
Recall:

\[ \phi_g(f) := \int fg \, d\mu \]

Theorem

For \( p \in (1, \infty) \), the map \( g \mapsto \phi_g \) is a linear bijection \( L^q \to (L^p)^* \) and is an isometry,

\[ \forall g \in L^q : \| \phi_g \| = \| g \|_q. \]

If \( \mu \) is \( \sigma \)-finite, the same holds also for \( p = 1 \).
Proof for $1 < p < \infty$

Lemma (Differentiability of $L^p$-norms)

For each $p \in (1, \infty)$ and each $f, h \in L^p$,

$$\frac{d}{dt} \|f + th\|^p_p \bigg|_{t=0} = p \int |f|^{p-2} fh \, d\mu$$

Proof: $F(t) := |f + th|^p$ convex and $C^1$,

$$F'(t) = p|f + th|^{p-2}[f + th]h$$

Convexity implies $t \mapsto F'(t)$ non-decreasing and $F'(0) \leq \frac{1}{t}[F(t) - F(0)] \leq F'(t)$ for any $t \geq 0$. Integrating over $\mu$ gives

$$p \int |f|^{p-2} fh \, d\mu \leq \frac{\|f + th\|^p_p - \|f\|^p_p}{t} \leq p \int |f + th|^{p-2}[f + th]h \, d\mu$$

Dominated Convergence permits taking $t \downarrow 0$. \qed
Let \( \phi \in (L^p)^* \) and set

\[
K := \{ h \in L^p : \phi(h) = 0 \}
\]

Pick any \( f \in L^p \setminus K \) (WLOG \( \phi \not= 0 \)). Then any \( f' \in L^p \) is a linear combination of \( f \) and an element from \( K \):

\[
f' = \frac{\phi(f')}{\phi(f)} f + h \quad \text{where} \quad h \in K
\]

Note that if \( f' = af + h \) then \( \phi(f') = a\phi(f) \) so \( \phi \) determined by its value at \( f \). Key problem: choose \( f \) in an optimal way.
Pick any \( f_0 \in L^p \setminus K \) with \( \phi(f_0) > 0 \). As \( K \) closed and convex, there is \( h_0 \in K \) such that

\[
\|f_0 - h_0\|_p = \inf_{h \in K} \|f_0 - h\|_p.
\]

Define \( f := f_0 - h_0 \) and note that \( \phi(f) = \phi(f_0) > 0 \) and

\[
\forall h \in K: \quad \int |f|^{p-2}fh \, d\mu = 0.
\]

Claim:

\[
\phi(f) = \|\phi\| \|f\|_p.
\]

Indeed, if \(< \) holds then \( \exists f' \in L^p \) with \( \phi(f') = \phi(f) \) and \( \|f'\|_p < \|f\|_p \). But then \( f' = f_0 - h \) for some \( h \in K \) and so \( \|f_0 - h\|_p < \|f_0 - h_0\|_p \), contradiction!
Define
\[ g(x) := \frac{\|\phi\|}{\|f\|_p^{p-1}} f(x) |f(x)|^{p-2} \]

Then \( g \in L^p \) and \( \phi_g(f) = \|\phi\| \|f\|_p \). Identity above gives \( \phi_g(h) = 0 \) for \( h \in K \), so \( \phi_g = \phi \).

For injectivity, note that, since \( q(p-1) = p \),
\[ \|g\|_q = \frac{\|\phi\|}{\|f\|_p^{p-1}} \left( \int |f|^{q(p-1)} d\mu \right)^{1/q} = \|\phi\| \]

and so \( g \mapsto \phi_g \) is a bijective isometry. \( \square \)
Proof for $p = 1$

Assume $\mu$ finite and let $\phi \in (L^1)^*$. Then $L^2 \subseteq L^1$ by
\[ \|f\|_1 \leq \mu(X)^{1/2}\|f\|_2 \] and so
\[ \forall f \in L^2 : \quad |\phi(f)| \leq \|\phi\|\mu(X)^{1/2}\|f\|_2. \]

By Theorem for $p = 2$ we get
\[ \forall f \in L^2 : \quad \phi(f) = \int fg \, d\mu. \]

Taking $f := 1_A$ with $A := \{|g| > \|\phi\| + \delta\}$ we get $\|g\|_\infty \leq \|\phi\|$ so $g \in L^\infty$.

If $f \in L^1$ then $f_n := f1_{\{|f| \leq n\}} \rightarrow f$ in $L^1$ and so $\phi(f_n) \rightarrow \phi(f)$. As $f_n \in L^2$ and $\int f_n g d\mu \rightarrow \int fg d\mu$ we get $\phi(f) = \int fg d\mu$ for all $f \in L^1$.

Hölder: $|\phi(f)| \leq \|g\|_\infty\|f\|_1$ and so $\|\phi\| \leq \|g\|_\infty$. \qed
Alternative line of proof

Alternative argument via Radon-Nikodym Theorem.

Suppose $\mu$ finite, $p = 1$. Let $\phi \in (L^1)^*$. Then

$$\nu(A) := \phi(1_A), \quad A \in \mathcal{F}$$

a finite measure. As $|\nu(A)| \leq \|\phi\| \mu(A)$, By Radon-Nikodym, there is $g : X \to \mathbb{R}$ s.t.

$$\phi(1_A) = \int g1_A d\mu$$

Elementary estimates: $\|g\|_{\infty} \leq \|\phi\|$.

Linearity+$L^1$-convergence, extends from $1_A$ to $f \in L^1$.

Extends to $\sigma$-additive for $p = 1$.

Extension needed for $p > 1$ w/o finiteness on $\mu$
Lemma

Suppose \( \exists A \in \mathcal{F} \) with no non-empty measurable subsets of finite measure. Then \( \phi_{1_A} = 0 \) yet \( 1_A \in L^\infty \) is non-zero. Thus \( g \mapsto \phi_g \) as a map \( L^\infty \to (L^1)^* \) is not injective and, in particular, not isometric.

Proof: If \( L^1 = \{0\} \) then \((L^1)^* = \{0\} \) so assume \( \exists f \in L^1 \). Then \( \mu(|f| > \epsilon) < \infty \) for each \( \epsilon > 0 \) and so

\[
\forall f \in L^1: \quad \mu(A \cap \{f \neq 0\}) = 0
\]

This means \( \phi_{1_A} = 0 \) yet \( 1_A \neq 0 \) because \( \mu(A) > 0 \). \( \square \)
Lemma

Let $X$ be an uncountable set, $\mathcal{F} = 2^X$ and $\mu$ the counting measure. Let $\mathcal{F}_0$ be the $\sigma$-algebra of countable and co-countable sets and $\mu_0$ the counting measure on $\mathcal{F}_0$. Then

$$L^1(\mu) = L^1(\mu_0) \land L^1(\mu)^* = L^1(\mu_0)^* = L^\infty(\mu) \neq L^\infty(\mu_0)$$

So $g \mapsto \phi_g$ is not surjective as the map $L^\infty(\mu_0) \to L^1(\mu_0)^*$.

Proof: If $f \in L^1(\mu)$ then $\{f \neq 0\}$ countable so $f \in L^1(\mu_0)$. Hence $L^1(\mu_0) = L^1(\mu)$.

$L^\infty(\mu)$ contains all bounded functions, $L^\infty(\mu_0)$ only those that are constant outside a countable set. So $L^\infty(\mu) \neq L^\infty(\mu_0)$. \qed
Situation in $L^\infty$

g \mapsto \phi_g as a map $L^1 \to (L^\infty)^*$ is an (injective) isometry

Not surjective whenever $X$ partitions into infinitely many sets of positive measure. Need Hahn-Banach Theorem, to be discussed later.
Weak convergence

Idea: A subspace of $C(X)$ induces a "new" topology on $X$

Definition (Weak topology)

Let $\mathcal{V}$ be a normed space over $\mathbb{R}$ and let $\mathcal{V}^*$ be the space of continuous linear functionals on $\mathcal{V}$. The coarsest topology containing

$$\{\phi^{-1}(O) : O \subseteq \mathbb{R} \land \phi \in \mathcal{V}^*\}$$

is called the weak topology on $\mathcal{V}$.

Generally not first countable and so not metrizable. Nets are needed to describe convergence! Still:

Definition (Weakly convergent sequences)

$\{x_n\}_{n \geq 1} \subseteq \mathcal{V}$ is weakly convergent to $x \in \mathcal{V}$ (denoted $x_n \xrightarrow{\text{w}} x$) if

$$\forall \phi \in \mathcal{V}^* : \phi(x) = \lim_{n \to \infty} \phi(x_n)$$
Example

Define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) := \begin{cases} 1, & \text{if } [2^n x] \text{ is even}, \\ 0, & \text{else}. \end{cases}$$

Then $f_n \in L^p$ for all $1 \leq p < \infty$ and so $\forall g \in L^q$:

$$\int_{[0,1]} g f_n \, d\lambda \xrightarrow{n \to \infty} \frac{1}{2} \int_{[0,1]} g \, d\lambda$$

So we think

$$\forall p \in [1, \infty) : \ f_n \xrightarrow{w} \frac{1}{2} 1_{[0,1]} \quad \text{in } L^p$$

Note: $\{f_n\}_{n \geq 1}$ not convergent in $L^p$ for any $p$. Weak convergence fails for $p = \infty$ as $\exists \phi \in (L^\infty)^* \text{ with } \phi(f_{2n}) = 1$ and $\phi(f_{2n+1}) = 0$. 
Q: Why is the limit unique?
A: Because weak-topology is Hausdorff!

Lemma (Continuous linear functionals on $L^p$ separate)

Let $p \in [1, \infty)$. Then

$$\forall f \in L^p \setminus \{0\} \exists \phi \in (L^p)^*: \phi(f) \neq 0$$

If $\mu$ is semifinite, then same true for $p = \infty$.

Proof: For $1 \leq p < \infty, f \in L^p \setminus \{0\}$ let $g := |f|^{p-2}f$. Then $g \in L^q$ and $\phi_g(f) = \|f\|_p^p > 0$.

For $p = \infty$, if $f \in L^\infty \setminus \{0\}$ then, by semifinitness, $\exists \epsilon > 0$ and $\exists A \subseteq \{|f| > \epsilon\}$ with $\mu(A) \in (0, \infty)$. Now take $g := \text{sign}(f)1_A$. □

Note: True for all Banach spaces by Hahn-Banach theorem!
Q: Are weakly convergent sequences bounded?

Definition
A set $A \subseteq L^p$ is weakly bounded if

$$\phi \in (L^p)^* : \sup_{f \in A} |\phi(f)| < \infty.$$ 

By

$$\sup_{f \in A} |\phi(f)| \leq \|\phi\| \sup_{f \in A} \|f\|_p$$

if $A \subseteq L^p$ is norm-bounded, then it is weakly bounded.

For converse we need …
Theorem

Let \( p \in [1, \infty) \). Then for all non-empty \( A \subseteq L^p \),

\[
\left( \forall \phi \in (L^p)^* : \sup_{f \in A} |\phi(f)| < \infty \right) \Rightarrow \sup_{f \in A} \|f\|_p < \infty
\]

In particular, every weakly bounded subset of \( L^p \) is bounded in \( L^p \). The same holds for \( p = \infty \) whenever \( \mu \) is semifinite.

Note: True for all Banach spaces. Many different proofs (Baire-Category Theorem, or direct arguments)
Proof of Uniform boundedness principle

Will use the “sliding hump” argument. Let $p \in [1, \infty)$ and suppose $\sup_{f \in A} \|f\|_p = \infty$. Then $\exists \{f_n\}_{n \geq 1} \subseteq A$ s.t.

$$\lim_{n \to \infty} 3^{-n} \|f_n\|_p = \infty$$

Set

$$g_n := \frac{1}{\|f_n\|_p^{p-1}} |f_n|^{p-2} f_n$$

Then $\phi_{g_n} \in (L^p)^*$ with $\|\phi_{g_n}\| = \|g_n\|_q = 1$. Next define $\{\sigma_n\}_{n \geq 1} \in \{-1, +1\}^\mathbb{N}$ recursively by $\sigma_1 := +1$ and

$$\forall n \geq 2: \quad \sigma_n \phi_{g_n}(f_n) \left( \sum_{k=1}^{n-1} 3^{-k} \sigma_k \phi_k(f_n) \right) \geq 0$$

Set

$$\phi := \sum_{n \geq 1} 3^{-n} \sigma_n \phi_{g_n}$$

and estimate ...
|φ(f_n)| ≥ |3^{-n} \sigma_n \phi_{g_n}(f_n) + \sum_{k=1}^{n-1} 3^{-k} \sigma_k \phi_{g_k}(f_n)| - \left| \sum_{k>n} 3^{-k} \sigma_k \phi_{g_k}(f_n) \right|

≥ 3^{-n} |\phi_{g_n}(f_n)| - \sum_{k>n} 3^{-k} \|\phi_{g_k}\| \|f_n\|_p

= 3^{-n} \left( |\phi_{g_n}(f_n)| - \frac{1}{2} \|f_n\|_p \right)

From \phi_{g_n}(f_n) = \|f_n\|_p we get |φ(f_n)| ≥ \frac{1}{2} 3^{-n} \|f_n\|_p \to \infty !!!

For p = \infty use semifiniteness of \mu to find

A_n \subseteq \{|f_n| > \frac{2}{3} \|f_n\|_\infty \}

with \mu(A_n) \in (0, \infty). Then set g_n := \frac{1}{\mu(A_n)} 1_{A_n} \in L^1 and note that

\|\phi_{g_n}\| = 1 and \phi_{g_n}(f_n) ≥ \frac{2}{3} \|f_n\|_\infty. Now continue as before. \qed
Corollary

Assume $p \in [1, \infty)$ or $p = \infty$ with the underlying measure semifinite and let $\{\phi_{i,j} : i, j \geq 1\} \subseteq (L^p)^*$ be such that

$$\forall i \geq 1 \ \exists f_i \in L^p : \sup_{j \geq 1} |\phi_{i,j}(f_i)| = \infty$$

Then

$$\exists f \in L^p \ \forall i \geq 1 : \sup_{j \geq 1} |\phi_{i,j}(f)| = \infty$$

Proof: homework
Corollary

Assume $1 < p < \infty$ and let $\{\phi_n\}_{n \geq 1} \in (L^p)^*$ be such that

$$\forall f \in L^p : \quad \phi(f) := \lim_{n \to \infty} \phi_n(f) \text{ exists in } \mathbb{R}$$

Then $\phi \in (L^p)^*$.

Proof: NTS continuity.

Key fact: For $1 < p < \infty$, double dual $(L^p)^{**}$ isometric to $L^p$ and evaluation map $\phi \mapsto \phi(f)$ member of $(L^p)^*$ by $|\phi(f)| \leq \|\phi\| \|f\|_p$

Assumption gives: $\{\phi_n\}_{n \geq 1}$ weakly bounded in $(L^p)^*$

Uniform boundedness principle: $c := \sup_{n \geq 1} \|\phi_n\| < \infty$ and so $|\phi(f)| \leq c\|f\|_p$ which gives $\phi \in (L^p)^*$. \qed
Reflexive space

We used that $L^p$, for $1 < p < \infty$, adheres to:

Definition
A normed vector space $\mathcal{V}$ is reflexive if the evaluation map $x \mapsto x^{**}$ defined by

$$\forall \phi \in \mathcal{V}^*: \quad x^{**}(\phi) := \phi(x)$$

images $\mathcal{V}$ onto the double dual $\mathcal{V}^{**}$.

Note $x^{**}$ always injective; in fact, isometry by $|x^{**}(\phi)| \leq \|x\| \|\phi\|$

$L^1$ and $L^\infty$ are NOT reflexive in general!