Duality and weak convergence

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Outline:

- Continuous linear functionals over $L^p$
- Duality of $L^p$ with $L^q$ for $1 \leq p < q$
- Weak topology and convergence
Natural idea: if $X$ topological space, study $C(X)$

If $X =$ vector space, then consider linear functions

Definition (Continuous linear functionals)
A functional $\phi : L^p \to \mathbb{R}$ is said to be

1. *linear* if it respects the linear structure of $L^p$, i.e.,
   \[
   \forall f, g \in L^p \forall a, b \in \mathbb{R} : \phi(af + bg) = a\phi(f) + b\phi(g).
   \]

2. *continuous* if $\phi^{-1}(O)$ is open in $L^p$ for every open $O \subseteq \mathbb{R}$.

Notation: $(L^p)^* :=$ set of continuous linear functionals on $L^p$
Consider a measure space \((X, \mathcal{F}, \mu)\) and let \(p, q \in [1, \infty]\) be Hölder conjugate indices, i.e., \(p^{-1} + q^{-1} = 1\). For any \(g \in L^q\) set

\[
\phi_g(f) := \int fg \, d\mu
\]

Then \(\phi_g \in (L^p)^*\); i.e., \(\phi_g\) is a continuous linear functional on \(L^p\).

**Proof:** Hölder gives

\[
\left| \phi_g(f) \right| \leq \|f\|_p \|g\|_q
\]

so integral well defined for all \(f \in L^p\). Linearity clear. For continuity, note that \(\phi_g\) images \(B_X(f, r)\) into \(B_{\mathbb{R}}(\phi_g(f), r\|g\|_q)\). \(\square\)
Definition (Bounded linear functional)

A linear functional $\phi : L^p \to \mathbb{R}$ is bounded if

$$\exists c \in [0, \infty) \ \forall f \in L^p : \ |\phi(f)| \leq c \|f\|_p.$$ 

We then observe:

Lemma

For any linear functional $\phi : L^p \to \mathbb{R}$:

$\phi$ is continuous $\iff$ $\phi$ is bounded

Proof: $\Leftarrow$ proved above. For $\Rightarrow$ note that $\phi^{-1}((-1, 1))$ contains 0 and thus $B_X(0, r)$ for some $r > 0$. Then $\|f\|_p < r$ implies $|\phi(f)| \leq 1$ and, by homogeneity, $|\phi(f)| \leq r^{-1}\|f\|_p$. □
Proposition

\((L^p)^*\) is a linear vector space with addition and scalar multiplication defined by \((\phi + \psi)(f) := \phi(f) + \psi(f)\) and \((a\phi)(f) := a\phi(f)\).

Moreover, denoting

\[
\|\phi\| := \sup_{f \in L^p \setminus \{0\}} \frac{|\phi(f)|}{\|f\|_p}
\]

defines a norm on \((L^p)^*\). The space \((L^p)^*\) is complete in this norm.
Proof of Proposition

Linearity and properties of norm checked readily, so main task is to show completeness.

Suppose \( \{ \phi_n \}_{n \geq 1} \subseteq (L^p)^* \) Cauchy in norm \( \| \cdot \| \). Linearity implies

\[
\forall f \in L^p : \quad |\phi_n(f) - \phi_m(f)| = |(\phi_n - \phi_m)(f)| \leq \| \phi_n - \phi_m \| \| f \|_p
\]

so, for each \( f \in L^p \), the sequence \( \{ \phi_n(f) \}_{n \geq 1} \) is Cauchy in \( \mathbb{R} \). Set

\[
\phi(f) := \lim_{n \to \infty} \phi_n(f)
\]

Then \( \phi \) linear and obeys

\[
|\phi(f)| \leq c\|f\|_p
\]

with \( c := \lim_{n \to \infty} \| \phi_n \| \). So \( \phi \) bounded and so continuous. It remains to show \( \| \phi_n - \phi \| \to 0 \) as \( n \to \infty \) . . .
\[ |\phi(f) - \phi_n(f)| = \lim_{m \to \infty} |\phi_m(f) - \phi_n(f)| \leq \lim_{m \to \infty} \|\phi_m - \phi_n\| \|f\|_p \]

So
\[ \|\phi - \phi_n\| \leq \lim_{m \to \infty} \|\phi_m - \phi_n\| \]

Taking \( n \to \infty \), the RHS tends to zero by the assumed Cauchy property of \( \{\phi_n\}_{n \geq 1} \). Hence \( \phi_n \to \phi \) in norm \( \| \cdot \| \)

Note: noting specific to \( L^p \) above, works for all normed spaces!
Riesz representation theorem

Recall:

\[ \phi_g(f) := \int fg \, d\mu \]

Theorem

For \( p \in (1, \infty) \), the map \( g \mapsto \phi_g \) is a linear bijection \( L^q \to (L^p)^\ast \) and is an isometry,

\[ \forall g \in L^q: \| \phi_g \| = \|g\|_q. \]

If \( \mu \) is \( \sigma \)-finite, the same holds also for \( p = 1 \).
Lemma (Differentiability of $L^p$-norms)

For each $p \in (1, \infty)$ and each $f, h \in L^p$,

$$\frac{d}{dt} \left\| f + th \right\|_p^p \bigg|_{t=0} = p \int |f|^{p-2} fh \, d\mu$$

Proof: $F(t) := |f + th|^p$ convex and $C^1$,

$$F'(t) = |f + th|^{p-2}[f + th]h$$

Convexity implies $t \mapsto F'(t)$ non-decreasing and $F'(0) \leq \frac{1}{t} [F(t) - F(0)] \leq F'(t)$ for any $t \geq 0$. Integrating over $\mu$ gives

$$p \int |f|^{p-2} fh \, d\mu \leq \frac{\|f + th\|_p^p - \|f\|_p^p}{t} \leq p \int |f + th|^{p-2}[f + th]h \, d\mu$$

Dominated Convergence permits taking $t \downarrow 0$. □
Let $\phi \in (L^p)^*$ and set

$$K := \{ h \in L^p : \phi(h) = 0 \}$$

Pick any $f \in L^p \setminus K$ (WLOG $\phi \neq 0$). Then any $f' \in L^p$ is a linear combination of $f$ and an element from $K$:

$$f' = \frac{\phi(f')}{\phi(f)} f + h \quad \text{where} \quad h \in K$$

Note that if $f' = af + h$ then $\phi(f') = a\phi(f)$ so $\phi$ determined by its value at $f$. Key problem: choose $f$ in an optimal way.
Proof for $1 < p < \infty$ continued ...

Pick any $f_0 \in L^p \setminus K$ with $\phi(f_0) > 0$. As $K$ closed and convex, there is $h_0 \in K$ such that

$$\|f_0 - h_0\|_p = \inf_{h \in K} \|f_0 - h\|_p.$$ 

Define $f := f_0 - h_0$ and note that $\phi(f) = \phi(f_0) > 0$ and

$$\forall h \in K: \int |f|^{p-2}fh \, d\mu = 0.$$ 

Claim:

$$\phi(f) = \|\phi\| \|f\|_p.$$ 

Indeed, if $\prec$ holds then $\exists f' \in L^p$ with $\phi(f') = \phi(f)$ and $\|f'\|_p < \|f\|_p$. But then $f' = f_0 - h$ for some $h \in K$ and so $\|f_0 - h\|_p < \|f_0 - h_0\|_p$, contradiction!
Define
\[ g(x) := \frac{\|\phi\|}{\|f\|_p^{p-1}} f(x)|f(x)|^{p-2} \]

Then \( g \in L^q \) and \( \phi_g(f) = \|\phi\| \|f\|_p \). Identity above gives \( \phi_g(h) = 0 \) for \( h \in K \), so \( \phi_g = \phi \).

For injectivity, note that, since \( q(p - 1) = p \),

\[ \|g\|_q = \frac{\|\phi\|}{\|f\|_p^{p-1}} \left( \int |f|^{q(p-1)} d\mu \right)^{1/q} = \|\phi\| \]

and so \( g \mapsto \phi_g \) is a bijective isometry. \( \square \)
Assume $\mu$ finite and let $\phi \in (L^1)^*$. Then $L^2 \subseteq L^1$ by
\[\|f\|_1 \leq \mu(X)^{1/2}\|f\|_2\] and so
\[\forall f \in L^2: \quad |\phi(f)| \leq \|\phi\|\mu(X)^{1/2}\|f\|_2.\]

By Theorem for $p = 2$ we get
\[\forall f \in L^2: \quad \phi(f) = \int fg \, d\mu.\]

Taking $f := 1_A$ with $A := \{|g| > \|\phi\| + \delta\}$ we get $\|g\|_\infty \leq \|\phi\|$ so $g \in L^\infty$.

If $f \in L^1$ then $f_n := f 1_{\{|f| \leq n\}} \to f$ in $L^1$ and so $\phi(f_n) \to \phi(f)$. As $f_n \in L^2$ and $\int f_n g \, d\mu \to \int fg \, d\mu$ we get $\phi(f) = \int fg \, d\mu$ for all $f \in L^1$.

Hölder: $|\phi(f)| \leq \|g\|_\infty \|f\|_1$ and so $\|\phi\| \leq \|g\|_\infty.$
Counterexamples for $L^1$

Lemma

Suppose $\exists A \in \mathcal{F}$ with no non-empty measurable subsets of finite measure. Then $\phi_{1_A} = 0$ yet $1_A \in L^\infty$ is non-zero. Thus $g \mapsto \phi_g$ as a map $L^\infty \to (L^1)^*$ is no injective and, in particular, not isometric.

Proof: If $L^1 = \{0\}$ then $(L^1)^* = \{0\}$ so assume $\exists f \in L^1$. Then $\mu(|f| > \epsilon) < \infty$ for each $\epsilon > 0$ and so

$$\forall f \in L^1: \quad \mu(A \cap \{f \neq 0\}) = 0$$

This means $\phi_{1_A} = 0$ yet $1_A \neq 0$ because $\mu(A) > 0$. \qed
Lemma

Let $X$ be an uncountable set, $\mathcal{F} = 2^X$ and $\mu$ the counting measure. Let $\mathcal{F}_0$ be the $\sigma$-algebra of countable and co-countable sets and $\mu_0$ the counting measure on $\mathcal{F}_0$. Then

$$L^1(\mu) = L^1(\mu_0) \land L^1(\mu)^* = L^1(\mu_0)^* = L^\infty(\mu) \neq L^\infty(\mu_0)$$

So $g \mapsto \phi_g$ is not surjective as the map $L^\infty(\mu_0) \to L^1(\mu_0)^*$.

Proof: If $f \in L^1(\mu)$ then $\{f \neq 0\}$ countable so $f \in L^1(\mu_0)$. Hence $L^1(\mu_0) = L^1(\mu)$.

$L^\infty(\mu)$ contains all bounded functions, $L^\infty(\mu_0)$ only those that are constant outside a countable set. So $L^\infty(\mu) \neq L^\infty(\mu_0)$.  \qed
$g \mapsto \phi_g$ as a map $L^1 \to (L^\infty)^*$ injective isometry

Not surjective whenever $X$ partitions into infinitely many sets of positive measure. Need Hahn-Banach Theorem, to be discussed later.