Analytic sets

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Plan:
- Continuous Lebesgue non-measurable function
- Analytic sets: 3 basic characterizations
- Analytic vs Borel, Lusin’s separation theorem
- Universal analytic set
- Universal measurability
Fact: \( f : \mathbb{R} \to \mathbb{R} \) continuous \( \Rightarrow \) Borel measurable

Question: Is \( f \) Lebesgue measurable?

Answered in:

**Lemma**

*Assuming the Axiom of Choice, there exists a continuous, strictly increasing \( f : \mathbb{R} \to \mathbb{R} \) that is not \( \mathcal{L}(\mathbb{R})/\mathcal{L}(\mathbb{R}) \)-measurable.*

Idea: By Banach-Zarecki, there is \( f \) continuous, strictly increasing s.t.

\[
\exists E \subseteq \mathbb{R}: \lambda(E) = 0 \land \lambda^*(f^{-1}(E)) > 0
\]

Now pick a non-measurable subset of \( E \). Need ...
Lemma

Assuming the Axiom of Choice, there exists $A \subseteq \mathbb{R}$ s.t.

$$\forall E \in \mathcal{L}(\mathbb{R}): \quad \lambda(E) = \lambda^*(A \cap E) = \lambda^*(E \setminus A).$$

In particular, $A \notin \mathcal{L}(\mathbb{R})$ and, in fact,

$$\forall E \in \mathcal{L}(\mathbb{R}): \quad \lambda(E) > 0 \quad \Rightarrow \quad E \cap A \notin \mathcal{L}(\mathbb{R}).$$
Construction of $A$

Pick $\alpha \notin \mathbb{Q}$ and set

$$x \sim y := \exists m, n \in \mathbb{Z} : x - y = m + n\alpha$$

Use AC to choose representatives using $\phi : \{[x] : x \in \mathbb{R}\} \to \mathbb{R}$ obeying $\phi([x]) \in [x]$ for all $x$. Then set

$$A := \{2m + n\alpha + \phi([x]) : x \in \mathbb{R} \land m, n \in \mathbb{Z}\}.$$ 

If $\lambda^*(E \setminus A) < \lambda(E)$ then find $B \in \mathcal{L} (\mathbb{R})$ with

$$E \setminus A \subseteq B \land \lambda^*(E \setminus A) = \lambda(B)$$

Now set $F := E \setminus B$ and note $F \subseteq A$.

As $\lambda(F) > 0$, $F - F$ contains open interval around 0. So does $A - A$ and, since $\{2m + 1 + n\alpha : m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$,

$$(A - A) \cap \{2m + 1 + n\alpha : m, n \in \mathbb{Z}\} \neq \emptyset$$
But $x \in A - A$ means $x = 2m' + n'\alpha + z - z'$ for some $z, z' \in \{\phi([y]): y \in \mathbb{R}\}$ and $x = 2m + 1 + n\alpha$ then forces $z \sim z'$ and, since each class represented uniquely, so $z = z'$. Thus

$$2m' + n'\alpha = 2m + 1 + n\alpha$$

As $\alpha \notin \mathbb{Q}$, this forces $n = n'$ and $2m' = 2m + 1$, which is absurd!

So $\lambda^*(E \setminus A) = \lambda(E)$. The argument for $E \cap A$ similar:

$$A' := \{2m + 1 + n\alpha + \phi([x]): x \in \mathbb{R} \land m, n \in \mathbb{Z}\}$$

obeys $A' = \mathbb{R} \setminus A$ and so $E \cap A = E \setminus A'$. ∎
Non-measurable function

\[ F := \text{Cantor function (a.k.a. Devil’s staircase) and } C := \text{Cantor ternary set. Define} \]
\[ G(x) := x + F(x) \]

Then \( \mu_G = \lambda + \mu_F \) and \( \mu_G(B) = \lambda(G(B)) \) shows \( \lambda(G(C)) = 1 \).

For \( A \) as above,
\[ A \cap G(C) \notin \mathcal{L}(\mathbb{R}) \]

yet
\[ B := G^{-1}(A \cap G(C)) \text{ obeys } \lambda(B) = 0 \]

so \( B \in \mathcal{L}(\mathbb{R}) \). For \( f := G^{-1} \) we get
\[ f^{-1}(B) = A \cap G(C) \notin \mathcal{L}(\mathbb{R}) \]
Measurability questions

The example shows \( \mathcal{B}(\mathbb{R}) \neq \mathcal{L}(\mathbb{R}) \) and motivates

\[
\mathcal{F} := \bigcap_{f: \mathbb{R} \to \mathbb{R} \text{ continuous}} \{ E \subseteq \mathbb{R} : f^{-1}(E) \in \mathcal{L}(\mathbb{R}) \}
\]

Then \( \mathcal{F} = \sigma\)-algebra s.t. all continuous functions are \( \mathcal{L}(\mathbb{R})/\mathcal{F} \)-measurable. We know

\[
\mathcal{B}(\mathbb{R}) \subseteq \mathcal{F} \subsetneq \mathcal{L}(\mathbb{R})
\]

but is \( \mathcal{F} \neq \mathcal{B}(\mathbb{R}) \)?

And what if \( \mathcal{L}(\mathbb{R}) \) is replaced by \( \mathcal{F} \); do we get a proper sub-\( \sigma \)-algebra of \( \mathcal{F} \)? And if so, what if we iterate?

And, for what \( \sigma \)-algebras \( \mathcal{F} \) is every continuous \( f: \mathbb{R} \to \mathbb{R} \) \( \mathcal{F}/\mathcal{F} \)-measurable?
Question: Is the continuous image of a $\mathcal{B}(\mathbb{R})$-set in $\mathcal{B}(\mathbb{R})$?

Lebesgue (1905): yes for projections of $\mathcal{B}(\mathbb{R}^2)$-sets onto $\mathbb{R}$

Suslin (1917): No! In fact, it must be of the form

$$\bigcup \bigcap_{\{n_i\}_{i \geq 1} \in \mathbb{N}^\omega \; k \geq 1} I_{n_1, \ldots, n_k}$$

where

$$\{I_{n_1, \ldots, n_k} : k \geq 1 \land \{n_i\}_{i \geq 1} \in \mathbb{N}^\omega\}$$

are closed intervals. Note: Uncountable union!
Definition

Let $X$ be a Polish space. A set $A \subseteq X$ is said to be *analytic* if it is a continuous image of a Polish space.

Not the original (Suslin’s) definition nor connected to Suslin’s form of a projections sets. We thus prove …
Theorem

*Every Polish space is a continuous image of the Baire space* $\mathbb{N}^\mathbb{N}$

Before giving the proof, note two Corollaries …
Corollary

Let $X$ be a Polish space and let $A \subseteq X$. The following are equivalent:

1. $A$ is analytic,
2. $A$ is a continuous image of $\mathbb{N}^\mathbb{N}$,
3. $A$ is the projection of a closed set in $X \times \mathbb{N}^\mathbb{N}$ onto $X$.

Proof:

(1) $\Rightarrow$ (2): $A = f(X)$ and, by Theorem, $X = g(\mathbb{N}^\mathbb{N})$, for $f, g$ cont.

(2) $\Rightarrow$ (3): Write $A = f(\mathbb{N}^\mathbb{N})$ and define $g: \mathbb{N}^\mathbb{N} \to X \times \mathbb{N}^\mathbb{N}$ by

$$g(\bar{n}) := (f(\bar{n}), \bar{n})$$

Then $g$ continuous and $g(\mathbb{N}^\mathbb{N})$ is closed and projects onto $A$

(3) $\Rightarrow$ (1): If $C \subseteq X \times \mathbb{N}^\mathbb{N}$ closed, then $C$ is a Polish space. The projection is continuous so definition applies.
Corollary

Every subset of $\mathbb{R}$ of the form

$$\bigcup_{\{n_i\}_{i \geq 1} \in \mathbb{N}^\mathbb{N}} \bigcap_{k \geq 1} I_{n_1, \ldots, n_k}$$

with

$$\{I_{n_1, \ldots, n_k} : k \geq 1 \land \{n_i\}_{i \geq 1} \in \mathbb{N}^\mathbb{N}\}$$

non-empty closed intervals is analytic.

Proof: $\{(x, \vec{n}) \in \mathbb{R} \times \mathbb{N}^\mathbb{N} : x \in I_{n_1, \ldots, n_k}\}$ is closed in $\mathbb{R} \times \mathbb{N}^\mathbb{N}$. Hence, so is

$$\bigcap_{k \geq 1} \left\{(x, \vec{n}) \in \mathbb{R} \times \mathbb{N}^\mathbb{N} : x \in I_{n_1, \ldots, n_k}\right\}.$$ 

This projects to the set above.
Proof of Theorem

WTS: \( \forall A \subseteq X \) analytic \( \exists f: \mathbb{N}^\mathbb{N} \to X \) continuous: \( A = f(\mathbb{N}^\mathbb{N}) \)

Let \( \{x_n\}_{n \geq 1} \) be dense in \( \mathbb{R} \) and \( B'(x, r) := \{y \in X: \varrho(x, y) \leq r\} \).

Define non-empty closed sets \( C_{n_1, \ldots, n_k} \subseteq X \) indexed by \( k \geq 1 \) and \( \{n_i\}_{i \geq 1} \in \mathbb{N}^\mathbb{N} \) recursively by:

- For \( k = 1 \), set \( C_n := B'(x_n, 1/2) \ \forall n \in \mathbb{N} \).
- If \( C_{n_1, \ldots, n_k} \) defined for some \( k \geq 1 \), enumerate

\[
\{m \in \mathbb{N}: B'(x_m, 2^{-k-1}) \cap C_{n_1, \ldots, n_k} \neq \emptyset\}
\]

into \( \{m_\ell\}_{\ell \geq 1} \) and set

\[
C_{n_1, \ldots, n_k, \ell} := C_{n_1, \ldots, n_k} \cap B'(x_{m_\ell}, 2^{-k-1})
\]

Then \( C_{n_1, \ldots, n_k} \neq \emptyset \) and closed. Moreover we get …
Proof of Theorem continued . . .

- Nesting and partitioning:
  \[ C_{n_1, \ldots, n_k} = \bigcup_{m \in \mathbb{N}} C_{n_1, \ldots, n_k, m} \]

- Diameter estimate
  \[ \text{diam}(C_{n_1, \ldots, n_k}) \leq 2^{-k+1} \]

It follows (HW problem)

\[ \bigcap_{k \geq 1} C_{n_1, \ldots, n_k} = \{y_{\bar{n}}\} \]

Claim: \( \tilde{n} \mapsto y_{\tilde{n}} \) continuous

Under \( d(\tilde{n}, \tilde{n}') := \sum_{k \geq 1} 2^{-k} 1_{\{n_k = n'_k\}} \), if \( 2^{-k-1} \leq d(\tilde{n}, \tilde{n}') < 2^{-k} \) then \( n_i = n'_i \) for \( i \leq k \). So \( y_{\tilde{n}}, y_{\tilde{n}'} \in C_{n_1, \ldots, n_k} \) and thus

\[ d(y_{\tilde{n}}, y_{\tilde{n}'}) \leq \text{diam}(C_{n_1, \ldots, n_k}) \leq 2^{-k+1} \leq 4d(\tilde{n}, \tilde{n}') \]

Map is surjective by partitioning property. \( \square \)
Properties of analytic sets

For analytic sets, we will now discuss:

- Behavior under image/preimage by continuous maps
- Containment of all closed and open sets
- Closeness under countable unions and intersections
- Inclusion of Borel sets
- Separation by Borel sets
Lemma

Let $X$ and $Y$ be Polish spaces and $f : X \to Y$ a continuous map. Then

\[ \forall A \subseteq X \text{ analytic: } f(A) \text{ analytic (in } Y) \]

and

\[ \forall B \subseteq Y \text{ analytic: } f^{-1}(B) \text{ analytic (in } X) \]
Proof of Lemma

Image map:
\( A \subseteq X \) analytic \( \Rightarrow A = g(\mathbb{N}^\mathbb{N}) \) for \( g: \mathbb{N}^\mathbb{N} \rightarrow X \) continuous. Then \( f(A) = f \circ g(\mathbb{N}^\mathbb{N}) \). As \( \mathbb{N}^\mathbb{N} \) is Polish, \( f(A) \) is analytic.

Preimage map:
Let \( B \subseteq Y \) analytic. Then \( \exists C \subseteq Y \times \mathbb{N}^\mathbb{N} \) closed that projects on \( B \). For \( f: X \rightarrow Y \), define \( h: Y \times \mathbb{N}^\mathbb{N} \rightarrow X \times \mathbb{N}^\mathbb{N} \) by

\[
h(x, \bar{n}) = (f(x), \bar{n})
\]

Then \( D := h^{-1}(C) \) is closed and projects on \( f^{-1}(A) \). So: \( f^{-1}(A) \) analytic. \( \square \)
Lemma

All closed and all open subsets of a Polish space are analytic

Proof: Closed subsets of a Polish space are Polish. Same for open sets (HW). Identity map shows they are analytic. □

Generalizes further:

Theorem (Alexandroff’s Theorem)

Every non-empty $G_\delta$-subset of a Polish space is Polish

Converse: A subset of a Polish is Polish iff it is $G_\delta$
Lemma

The class of analytic subsets of a Polish space is closed under countable unions and countable intersections.

Let \( \{A_n\}_{n \geq 1} \) be analytic in \( X \). Then \( \exists f_n : Y_n \to X : A_n = f(X_n) \)

Countable unions:
Set \( Y := \bigcup_{n \geq 1} (\{n\} \times Y_n) \) and define \( g : Y \to X \) by \( g(n, y) := f_n(y) \). Then \( g(Y) = \bigcup_{n \geq 1} A_n \) so union analytic.

Countable intersections:
Define \( f : \prod_{n \geq 1} Y_n \to X^\mathbb{N} \) by \( f(\{y_n\}_{n \geq 1}) := \{f_n(y_n)\}_{n \geq 1} \). Then \( f \) continuous with \( f(\prod_{n \geq 1} Y_n) = \prod_{n \geq 1} A_n \). So \( \prod_{n \geq 1} A_n \) analytic.

Next define \( g : X \to X^\mathbb{N} \) by \( g(x) = \{x\}_{n \geq 1} \). Then

\[
g^{-1}\left( \prod_{n \geq 1} A_n \right) = \bigcap_{n \geq 1} A_n
\]

and so \( \bigcap_{n \geq 1} A_n \) analytic.
Proposition

*All Borel sets in a Polish space are analytic*

Could use Borel hierarchy. Instead rely on:

Lemma (Monotone class lemma)

Let $X$ be a set and assume $\mathcal{M} \subseteq 2^X$ contains $X$ and $\emptyset$ and is closed under countable unions and countable intersections. Then

$$\mathcal{F} := \{A \in \mathcal{M} : A^c \in \mathcal{M}\}.$$  

is a $\sigma$-algebra.

Proof of Proposition: $\mathcal{M} := \{A \subseteq X : \text{analytic}\}$ is a monotone class containing all closed/open sets. $\mathcal{F}$ contains closed/open sets. So $\mathcal{B}(X) \subseteq \mathcal{F} \subseteq \mathcal{M}$. 

□
Question: How to characterize analytic sets that are Borel?

Theorem

Let $X$ be Polish and $A, A' \subseteq X$ are analytic with $A \cap A' = \emptyset$. Then $A$ and $A'$ are Borel separated in the sense that

$$\exists B \in \mathcal{B}(X): \quad A \subseteq B \land A' \subseteq B^c$$

In particular,

$$\forall A \subseteq X \text{ analytic}: \quad A \in \mathcal{B}(X) \iff A^c \text{ is analytic}$$
Proof of Lusin’s separation theorem

Key idea: Divide and conquer

Let $\mathcal{A} := \text{class of analytic sets}$. Note: If $\{E_i\}_{i \geq 1} \subseteq \mathcal{A}$ and $\{E'_i\}_{i \geq 1} \subseteq \mathcal{A}$ obey

$$\left( \bigcup_{i \geq 1} E_i \right) \cap \left( \bigcup_{j \geq 1} E'_j \right) = \emptyset.$$ 

then

$$\forall i, j \geq 1 \exists B_{i,j} \in \mathcal{B}(X): \quad E_i \subseteq B_{i,j} \land E'_j \subseteq B_{i,j}^c$$

implies $\bigcup_{i \geq 1} E_i \subseteq \bigcup_{i \geq 1} \bigcap_{j \geq 1} B_{i,j}$ and

$$\bigcup_{j \geq 1} E'_j \subseteq \bigcup_{i \geq 1} \bigcap_{j \geq 1} B_{i,j}^c \subseteq \bigcap_{i \geq 1} \bigcup_{j \geq 1} B_{i,j}^c = \left( \bigcup_{i \geq 1} \bigcap_{j \geq 1} B_{i,j} \right)^c$$

Conclusion: If $\bigcup_{i \geq 1} E_i$ and $\bigcup_{j \geq 1} E'_j$ are NOT Borel separated, then at least one pair of sets $E_i$ and $E'_j$ is NOT Borel separated
Assume $A, A'$ disjoint analytic but NOT Borel separated.

Projection map: $f : X \times \mathbb{N}^\mathbb{N} \to X$. Then $\exists C, C' \subseteq X \times \mathbb{N}^\mathbb{N}$ closed, disjoint with $A = f(C)$ and $A' = f(C')$

Apply “divide and conquer” for intersections with closed balls of radii $2^{-k-1}, k \geq 1$, to recursively define $\{C_k\}_{k \geq 1}$ and $\{C'_k\}_{k \geq 1}$ closed non-empty such that (for all $k \geq 1$)

- $C_1 := C \land C'_1 := C'$
- $C_{k+1} \subseteq C_k \land C'_{k+1} \subseteq C'_k$
- $\text{diam}(C_k) \leq 2^{-k}$ and $\text{diam}(C'_k) \leq 2^{-k}$
- $f(C_k), f(C'_k)$ analytic but NOT Borel separated

But then $\bigcap_{k \geq 1} C_k = \{x\}$ and $\bigcap_{k \geq 1} C'_k = \{x'\}$ with $f(x) \in A$ and $f(x') \in A'$. So $x \neq x'$ and $C_k$ and $C'_k$ are separated by disjoint open balls. Hence, $f(C_k)$ and $f(C'_k)$ ARE Borel separated, a contradiction! \[\square\]
Next question: Are there analytic sets that are not Borel?
Answer: YES but we need to work a bit

Definition (Universal set)
Let $X$ be a set and $A$ a collection of its subsets. We say that a set $E \subseteq X \times \mathbb{N}^\mathbb{N}$ is universal for $A$ if

$$\forall A \in A \; \exists \bar{n} \in \mathbb{N}^\mathbb{N} : \{ x \in X : (x, \bar{n}) \in E \} = A.$$ 

In other words, each $A \in A$ appears as a section of $E$ of constant second coordinate.
Lemma

Let \((X, \mathcal{T})\) be a second-countable topological space. Then there exists an open set \(E \subseteq X \times \mathbb{N}^\mathbb{N}\) that is universal for \(\mathcal{T}\).

Proof:
Let \(\{U_n\}_{n \geq 1}\) be open sets generating \(\mathcal{T}\) by unions. Abbreviating \(\check{n} = \{n_i\}_{i \geq 1}\), let

\[
E := \left\{(x, \check{n}) \in X \times \mathbb{N}^\mathbb{N} : x \in \bigcup_{i \geq 1} U_{n_i}\right\}
\]

Open because \(\{(x, \check{n}) \in X \times \mathbb{N}^\mathbb{N} : x \in U_{n_i}\}\) is open. Let \(O \in \mathcal{T}\).
Then there is \(\check{n}' = \{n'_i\}_{i \geq 1} \in \mathbb{N}^\mathbb{N}\) such that \(O = \bigcup_{i \geq 1} U_{n'_i}\). So

\[
\left\{x \in X : (x, \check{n}') \in E\right\} = O
\]

and \(O\) is thus a section of \(E\). \(\square\)
Proposition

Let $X$ be a Polish space. Then there is an analytic set $E \subseteq X \times \mathbb{N}^\mathbb{N}$ that is universal for the class of analytic subsets of $X$.

Proof:
Let $F \subseteq (X \times \mathbb{N}^\mathbb{N}) \times \mathbb{N}^\mathbb{N}$ universal for open sets. Then $D := F^c$ universal for closed sets in $X \times \mathbb{N}^\mathbb{N}$
Next: $A \subseteq X$ analytic is a projection of a closed $C \subseteq X \times \mathbb{N}^\mathbb{N}$
Then $C$ is a section of $D$ for constant 3rd coordinate so letting

$$E := \text{projection of } D \text{ on 1st and 3rd coordinate}$$

we get that $A$ is a section of $E$. \qed
Corollary

The class of analytic (and thus also Borel) subsets of a Polish space is at most of the cardinality of the continuum. In any infinite Polish space $X$, the cardinality of $\mathcal{B}(X)$ is that of the continuum.

Proof:
Number of sections $\leq |\mathbb{N}^\mathbb{N}| = |\mathbb{R}|
X$ infinite $\Rightarrow \mathcal{B}(X)$ contains all subsets of an infinite set. So $|\mathcal{B}(X)| \geq |\{0, 1\}^\mathbb{N}| = |\mathbb{R}|$  

Note: This proves that $\mathcal{B}(\mathbb{R}) \not\subseteq \mathcal{L}(\mathbb{R})$
Lemma

The Baire space \( \mathbb{N}^\mathbb{N} \) contains an analytic set that is not Borel

Proof: Let \( E \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \) be universal for analytic sets in \( \mathbb{N}^\mathbb{N} \). Define

\[
A := \{ \bar{n} \in \mathbb{N}^\mathbb{N} : (\bar{n}, \bar{n}) \in E \}.
\]

Then \( A \) analytic. If \( A^c \) analytic, then

\[
\exists \bar{n}' \in \mathbb{N}^\mathbb{N} : A^c = \{ \bar{n} \in \mathbb{N}^\mathbb{N} : (\bar{n}, \bar{n}') \in E \}.
\]

As \( \bar{n}' \in A \) is equivalent to \( \bar{n}' \in A^c \). So \( A^c \) is NOT analytic and so \( A \not\in \mathcal{B}(\mathbb{N}^\mathbb{N}) \).
Theorem

Every uncountable Polish space $X$ contains an analytic set $A$ such that $A \notin \mathcal{B}(X)$

Proof: $X$ embeds $\{0, 1\}^{\mathbb{N}}$ which embeds $\mathbb{N}^{\mathbb{N}}$, with both maps homeomorphic and thus Borel-bimeasurable. The maps move the analytic non-Borel set in $\mathbb{N}^{\mathbb{N}}$ to $X$. 

□
The above motivates:

Definition (Co-analytic set)
A subset $A \subseteq X$ of a Polish space $X$ is said to be co-analytic if $A^c$ is analytic.

Interesting explicit examples:
- Mazurkiewicz 1936: Set of differentiable functions in $C([0, 1])$ co-analytic but not Borel
- Mauldin 1979: Set of nowhere differentiable functions in $C([0, 1])$ co-analytic but not Borel
Any measure $\mu$ extends uniquely to its completion $\bar{\mu}$ on:

Definition ($\mu$-measurable sets)

Let $\mu$ be a measure on a measurable space $(X, \mathcal{F})$. A set $A \subseteq X$ is then called $\mu$-measurable if

$$\exists E, F \in \mathcal{F} : \quad E \subseteq A \subseteq F \land \mu(F \setminus E) = 0.$$  

We will write $\mathcal{F}_\mu$ for the class of $\mu$-measurable sets.

The extended measure space $(X, \mathcal{F}_\mu, \bar{\mu})$ is complete.

Question: Characterize sets in $\mathcal{F}_\mu$
Theorem (Lusin)

Let $X$ be a Polish space and let $\mu$ be a finite measure on $(X, \mathcal{B}(X))$. Then every analytic or co-analytic subset of $X$ is $\mu$-measurable.
Outer measure:

\[ \mu^*(A) := \inf \{ \mu(B) : B \in \mathcal{B}(X) \land A \subseteq B \} . \]

Key step: Inner regularity (of \( \mu^* \))

Lemma

*Let \( A \subseteq X \) be analytic with \( A \neq \emptyset \). Then*

\[ \forall \epsilon > 0 \ \exists K \subseteq X : \ K \text{ compact} \land K \subseteq A \land \mu^*(A) \leq \mu(K) + \epsilon . \]
Proof of Lemma

Note: $\mu^*$ is regular so each set admits measurable cover. Hence: Monotone Convergence (MCT) holds for $\mu^*$

Pick $A \subseteq X$ and find $f : \mathbb{N}\mathbb{N} \to X$ continuous with $f(\mathbb{N}\mathbb{N}) = A$

Let

$$R_k(r_1, \ldots, r_k) := \left\{ \{n_i\}_{i \geq 1} \in \mathbb{N}\mathbb{N} : (\forall i \leq k : n_i \leq r_i) \right\}$$

Note $R_{k+1}(r_1, \ldots, r_k, r) \uparrow R_k(r_1, \ldots, r_k)$ as $r \to \infty$. So, by MCT,

$$\mu^*(f(R_1(r))) \xrightarrow{r \to \infty} \mu^*(A)$$

and

$$\mu^*(f(R_{k+1}(r_1, \ldots, r_k, r))) \xrightarrow{r \to \infty} \mu^*(f(R_k(r_1, \ldots, r_k))).$$

So, given $\epsilon > 0$, there is $\{r_i\}_{i \geq 1} \in \mathbb{N}\mathbb{N}$ s.t.

$$\forall k \geq 1 : \mu^*(f(R_k(r_1, \ldots, r_k))) \geq \mu^*(A) - \epsilon$$

Now define ...
... define

$$\tilde{K} := \left\{ \{n_i\}_{i \geq 1} \in \mathbb{N}^\mathbb{N} : (\forall i \geq 1: n_i \leq r_i) \right\}$$

Then $K := f(\tilde{K})$ is compact with $K \subseteq f(\mathbb{N}^\mathbb{N}) = A$. NTS:

$$K = \bigcap_{k \geq 1} f\left(R_k(r_1, \ldots, r_k)\right)$$

as that implies $\mu(K) \geq \mu^*(A) - \epsilon$ by MCT.

$\subseteq$ immediate. For $\supseteq$ pick $x$ in intersection. Then

$$\forall k \geq 1 \ \exists \tilde{n}^{(k)} \in R_k(r_1, \ldots, r_k): f(\tilde{n}^{(k)}) = x$$

But then $\exists k_j \to \infty$ s.t. $\tilde{n}^{(k_j)} \to \tilde{n} \in \tilde{K}$. Continuity:

$$x = f(\tilde{n}) \in f(\tilde{K}) = K$$
As $\mathcal{B}(X)_\mu$ $\sigma$-algebra, suffices to consider $A \subseteq X$ analytic.

Finite outer measure:

$$\forall \epsilon > 0 \exists B_\epsilon \in \mathcal{B}(X): A \subseteq B_\epsilon \land \mu(B) \leq \mu^*(A) + \epsilon$$

Lemma:

$$\forall \epsilon > 0 \exists K_\epsilon \in \mathcal{B}(X): K_\epsilon \subseteq A \land \mu(K_\epsilon) \geq \mu^*(A) - \epsilon$$

Take

$$E := \bigcup_{n \geq 1} K_{1/n} \land F := \bigcap_{n \geq 1} B_{1/n}$$

to get $E \subseteq A \subseteq F$ with $\mu(F \setminus E) = 0$. \qed
Definition (Universal measurability)

$A \subseteq X$ is *universally measurable* if it belongs to the $\sigma$-algebra

$$\mathcal{F}_\star := \bigcap_{\mu: \text{ finite measure on } (X, \mathcal{F})} \mathcal{F}_\mu$$

of universally measurable sets on $(X, \mathcal{F})$.

Lusin’s Theorem: For $(X, \mathcal{B}(X))$ with $X$ uncountable we have

$$\mathcal{B}(X) \subseteq \mathcal{B}(X)_\star$$

Important e.g. in stochastic analysis
Universally measurable functions

Definition

$f : X \to X$ is said to be universally measurable if $f^{-1}(\mathcal{F}_*) \subseteq \mathcal{F}_*$.

Lemma

Let $(X, \mathcal{F})$ be measurable space. Then every $\mathcal{F}/\mathcal{F}$-measurable $f : X \to X$ is universally measurable.
Analytic sets play important in logic, set up for certain probability models, etc

More info:
- A.S. Kechris’ “Classical descriptive set theory”
- D.L. Cohn’s “Measure theory”