Product spaces, disintegration, conditional measures

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Goal: Product measurable spaces and measures, over finite and infinite index sets, role of topology in handling these

Plan:
- Finitely-indexed product spaces, Fubini-Tonelli’s Theorem
- Disintegration and conditional measures

Building towards:
- Infinite product spaces and product measures
- Prescribing measures by finite-dimensional marginals, Kolmogorov’s Extension theorem
Finite product space

Fubini-Tonelli’s Theorem handles exchangeability of iterated integrals. Key notion:

Definition (Product measurable space)

Let \((X_1, \mathcal{F}_1), \ldots, (X_n, \mathcal{F}_n)\) be measurable spaces. The product measurable space is then the pair \((\times_{i=1}^n X_i, \otimes_{i=1}^n \mathcal{F}_i)\) where

\[
\otimes_{i=1}^n \mathcal{F}_i := \sigma \left( \left\{ \times_{i=1}^n A_i : (\forall i = 1, \ldots, n : A_i \in \mathcal{F}_i) \right\} \right) \tag{1}
\]

is the product \(\sigma\)-algebra.

Alternative notation \(\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n\)

Associative:

\[
(\mathcal{F}_1 \otimes \mathcal{F}_2) \otimes \mathcal{F}_3 = \mathcal{F}_1 \otimes (\mathcal{F}_2 \otimes \mathcal{F}_3) = \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3
\]
Measurability of sections

For $E \subseteq X \times Y$ and $x \in X$ define

$$E^x := \{y \in Y: (x, y) \in E\}$$

to be the section of $E$ of constant $x$-variable.

Lemma (Fubini-Tonelli Theorem, measurability part)

Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces. Then

$$\forall A \in \mathcal{F} \forall x \in X: A^x \in \mathcal{G}$$

and similarly for the section in the second variable.

Given a measure space $(Z, \mathcal{H})$ and an $\mathcal{F} \otimes \mathcal{G} / \mathcal{H}$-measurable function $f: X \times Y \to Z$, for any $x \in X$,

$$y \mapsto f(x, y)$$

is $\mathcal{G}$-measurable

and $x \mapsto f(x, y)$ is $\mathcal{F}$-measurable for all $y \in Y$. 


Proof of Lemma

The class
\[ \mathcal{C} := \{ C \in \mathcal{F} \otimes \mathcal{G} : C^x \in \mathcal{G} \} \]
is a \( \sigma \)-algebra. Since it contains the semi-algebra
\[ S := \{ A \times B : A \in \mathcal{F} \land B \in \mathcal{G} \} \]
it contains \( \sigma(S) = \mathcal{F} \otimes \mathcal{G} \).

Given \( f : X \times Y \to Z \) and \( x \in X \), let \( f_x(y) := f(x,y) \). Then
\[ \forall B \subseteq Z : f_x^{-1}(B) = f^{-1}(B)^x \]
and so if \( f \) is \( \mathcal{F} \otimes \mathcal{G}/\mathcal{H} \)-measurable, \( f_x \) is \( \mathcal{G}/\mathcal{H} \)-measurable. \( \square \)
Theorem (Product measure)

Let \((X, \mathcal{F}, \mu)\) and \((Y, \mathcal{G}, \nu)\) measure spaces. Then there exists a measure \(\mu \otimes \nu\) on \((X \times Y, \mathcal{F} \otimes \mathcal{G})\) such that

\[
\forall A \in \mathcal{F} \forall B \in \mathcal{G} : \mu \otimes \nu(A \times B) = \mu(A)\nu(B)
\]

where \(0 \cdot \infty := 0\). If both \(\mu\) and \(\nu\) are \(\sigma\)-finite, then \(\mu \otimes \nu\) is unique.
Proof of Theorem

Semialgebra of rectangles $S := \{A \times B: A \in \mathcal{F} \land B \in \mathcal{G}\}$. Set

$$\kappa(A \times B) := \mu(A)\nu(B)$$

Claim: $\kappa$ countably additive on $S$

Indeed, if $\{A_i \times B_i\}_{i \geq 1}$ are disjoint with $\bigcup_{i \geq 1} A_i \times B_i = A \times B$, then for all $x \in X$ and $y \in Y$,

$$1_A(x)1_B(y) = \sum_{i \geq 1} 1_{A_i}(x)1_{B_i}(y)$$

Now integrate over $x$ w.r.t. $\mu$ and then over $y$ w.r.t. $\nu$ to get, with the help of Monotone Convergence Theorem,

$$\mu(A)\nu(B) = \sum_{i \geq 1} \mu(A_i)\nu(A_i)$$

Since $S$ semi-algebra, $\kappa$ extends to a measure on $\sigma(S) = \mathcal{F} \otimes \mathcal{G}$. Under $\sigma$-finiteness, $\pi$-$\lambda$-Theorem implies uniqueness. \qed
General products

Note: Same holds for product of more spaces,

\[ \mu_1 \otimes \cdots \otimes \mu_n(A_1 \times \cdots \times A_n) = \prod_{i=1}^{n} \mu_i(A_i) \]

We call \( \mu_1 \otimes \cdots \otimes \mu_n \) the product measure.

The product is associative:

\[ (\mu_1 \otimes \mu_2) \otimes \mu_3 = \mu_1 \otimes (\mu_2 \otimes \mu_3) = \mu_1 \otimes \mu_2 \otimes \mu_3 \]
Theorem (Fubini-Tonelli)

Let \((X, \mathcal{F}, \mu)\) and \((Y, \mathcal{G}, \nu)\) be \(\sigma\)-finite measure spaces and let \(f : X \times Y \to \mathbb{R}\) be \(\mathcal{F} \otimes \mathcal{G} / \mathcal{B}(\mathbb{R})\)-measurable. Assume that either \(f \geq 0\) (Tonelli) or \(f \in L^1(\mu \otimes \nu)\) (Fubini).

For \(y \mapsto f(x, y)\) is integrable w.r.t. \(\nu\) for all \(x \in X\) and

\[
x \mapsto \int f(x, y) \, \nu(dy) \quad \text{is } \mathcal{F}\text{-measurable}
\]

and, similarly \(x \mapsto f(x, y)\) is integrable w.r.t. \(\mu\) for all \(y \in Y\) and \(y \mapsto \int f(x, y) \, \mu(dx)\) is \(\mathcal{G}\)-measurable. These functions are nonnegative (Tonelli) or in \(L^1(\mu)\), resp., \(L^1(\nu)\) (Fubini). Moreover,

\[
\int f \, d\mu \otimes \nu = \int \left( \int f(x, y) \, \nu(dy) \right) \mu(dx)
\]

\[
= \int \left( \int f(x, y) \, \mu(dx) \right) \nu(dy)
\]
Proof outline

- Tonelli holds for $f := 1_{A \times B}$.
- $\{C \in F \otimes G : \text{Tonelli holds for } 1_C\}$ is a $\sigma$-algebra, and so Tonelli holds for all $f := 1_C$ with $C \in F \otimes G$.
- Monotone Class Lemma $\Rightarrow$ Tonelli holds for all measurable $f \geq 0$
- Tonelli $\Rightarrow$ Fubini by $f = f^+ - f^-$
Important counterexamples

- $X = Y := \mathbb{N}, \mathcal{F} = \mathcal{G} := 2^\mathbb{N}, \mu = \nu := \text{counting measure},$
  
  \[ f(m, n) = 1_{\{n=m\}} - 1_{\{n=m-1\}} \]

  Then iterated integrals not equal. Reason: $f \notin L^1$

- $X = Y := \text{totally ordered uncountable s.t.} \ \{y \in X: y < x\}$
  countable $\forall x \in X, \mathcal{F} = \mathcal{G} := \text{countable/co-countable sets},$
  $\mu = \nu := 1 \text{ of co-countable sets, else } 0.$ Then

  \[ \int \left( \int 1_{\{x < y\}} \mu(dx) \right) \nu(dy) \neq \int \left( \int 1_{\{x < y\}} \nu(dy) \right) \mu(dy) \]

  Reason: $1_{\{x < y\}}$ not measurable

- $X = Y := [0, 1], \mathcal{F} := \mathcal{B}([0, 1]), \mathcal{G} := 2^Y, \mu := \text{Lebesgue measure, } \nu := \text{the counting measure and } f(x, y) := 1_{\{x=y\}}.$
  Iterated integrals not equal. Reason: $\nu$ not $\sigma$-finite
Disintegration

Turning Fubini-Tonelli’s problem around we ask to disintegrate a measure \( \kappa \) on \( (X \times Y, \mathcal{F} \otimes \mathcal{G}) \) as

\[
\int_{X \times Y} f \, d\kappa = \int_X \left( \int_Y f(x,y) \nu_x(\,dy) \right) \mu(\,dx)
\]

where \( \{\nu_x : x \in X\} \) is a family of measures.

For measures themselves we get:

\[
\forall A \in \mathcal{F} \forall B \in \mathcal{G} : \quad \kappa(A \times B) = \int_A \nu_x(B) \mu(\,dx)
\]

Note: Need \( x \mapsto \nu_x(B) \mathcal{F} \)-measurable for each \( B \in \mathcal{G} \). This suffices to ensure that

\[
x \mapsto \int_Y f(x,y) \nu_x(\,dy)
\]

is \( \mathcal{F} \)-measurable for all \( \mathcal{F} \otimes \mathcal{G} \)-measurable \( f \geq 0 \).
An almost solution

Natural normalization $\nu_x(Y) = 1$ implies

$$\mu(A) = \kappa(A \times Y)$$

Then $\kappa(\cdot \times B) \ll \mu$ and (under $\sigma$-finiteness) we solve this by

$$\kappa(A \times B) = \int_A (1_B)_F \, d\mu$$

Hence we get $\nu_x(B) = (1_B)_F$ for $\mu$-a.e. $x$.

Note: $\forall \{B_i\}_{i \geq 1} \subseteq F$ disjoint:

$$(1_{\bigcup_{i \geq 1} B_i})_F = \sum_{i \geq 1} (1_{B_i})_F \text{ $\mu$-a.e.}$$

Problem: Null set depends on $\{B_i\}_{i \geq 1}$!
Generalizing to conditional measures

Extend disintegration beyond product $\sigma$-algebras:

Definition (Conditional measures)
Let $\mu$ be a measure on $(X, \mathcal{F})$ and let $\mathcal{G} \subseteq \mathcal{F}$ be a $\sigma$-algebra. A family $\{\nu_x : x \in X\}$ such that

1. $\forall x \in X: \nu_x$ is a probability measure on $(X, \mathcal{F})$,
2. $\forall A \in \mathcal{F}: x \mapsto \nu_x(A)$ is $\mathcal{G}$-measurable and

$$\forall B \in \mathcal{G}:\; \mu(A \cap B) = \int_B \nu_x(A) \mu(dx)$$

are the conditional measures for $\mu$ given $\mathcal{G}$.

Note: $A = X \times A', B = B' \times Y$ implies $A \cap B = A' \times B'$. Role of $\mathcal{F}$ and $\mathcal{G}$ different than before!
Connection with conditional expectation

Recall: $f_G := \text{conditional expectation of } f \in L^1(\mu) \text{ given } \mathcal{G}$

Lemma

If conditional measures $\{\nu_x : x \in X\}$ for $\mu$ given $\mathcal{G}$ exist, then

$$\forall f \in L^1(\mu) : \quad f_G(x) = \int f \, d\nu_x \text{ for } \mu\text{-a.e. } x \in X.$$  

Proof: True for $f := 1_A$ for all $A \in \mathcal{F}$. Extension to $f \in L^1(\mu)$ by Monotone Class Lemma.

Upshot: Conditional expectation ($=\text{integral}$) is an integral.
Definition

$\mathcal{F}$ is said to be countably generated if $\exists \{A_n : n \geq 1\} \subseteq \mathcal{F}$ such that $\mathcal{F} = \sigma(\{A_n : n \geq 1\})$.

Lemma (Uniqueness a.e.)

Let $\mu$ be a finite measure on $(X, \mathcal{F})$ with $\mathcal{F}$ countably generated and let $\mathcal{G} \subseteq \mathcal{F}$ be a $\sigma$-algebra. For any conditional measures $\{\nu_x : x \in X\}$ and $\{\tilde{\nu}_x : x \in X\}$ given $\mathcal{G}$ there is $X^* \in \mathcal{G}$ such that

$$\mu(X \setminus X^*) = 0$$

and

$$\forall x \in X^* \forall A \in \mathcal{F} : \nu_x(A) = \tilde{\nu}_x(A).$$

In other words, the two families agree $\mu$-a.e.
Proof of uniqueness

\[ \mathcal{A} := \text{algebra generated by } \{\mathcal{A}_n : n \geq 1\}. \text{ Still countable. Set} \]
\[ X_A := \{x \in X : \nu_x(A) = \tilde{\nu}_x(A)\} \]

Then \( X_A \in \mathcal{G} \) and, since \( \nu_x(A) = (1_A)_\mathcal{G}(x) = \tilde{\nu}_x(A) \) for \( \mu \)-a.e. \( x \), \( \mu(X \setminus X_A) = 0 \). Hence
\[ X^* := \bigcap_{A \in \mathcal{A}} X_A \]
obeys \( X^* \in \mathcal{G} \) and \( \mu(X \setminus X^*) = 0 \). As
\[ \mathcal{L} := \{A \in \mathcal{F} : (\forall x \in X^* : \nu_x(A) = \tilde{\nu}_x(A))\} \]
is \( \lambda \)-system containing \( \pi \)-system \( \mathcal{A} \), we get \( \sigma(\mathcal{P}) \subseteq \mathcal{L} \subseteq \mathcal{F} \). \( \square \)
Conditional measures may not exist

Abbreviate $I := [0,1]$

Lemma

Set $\lambda_2 := \text{Lebesgue measure on } (X, \mathcal{F})$ where

$$X := I \times I \quad \text{and} \quad \mathcal{F} := \mathcal{L}(I \times I)$$

Let

$$\mathcal{G} := \{A \times I : A \subseteq \mathcal{B}(I)\}$$

Then, assuming the existence of Lebesgue non-measurable sets, there is no family of conditional measures for $\lambda_2$ given $\mathcal{G}$.

Use of $\mathcal{L}(I \times I)$ key! Conditional measures do exist for $\mathcal{B}(I \times I)$. 
Assume \( \{\nu_{(x,y)} : (x, y) \in I \times I\} \) be conditional measures given \( \mathcal{G} \). As \( \mathcal{G} \subseteq \mathcal{B}(I \times I) \subseteq \mathcal{L}(I \times I) \) these are also conditional measures for Lebesgue measure on \((I \times I, \mathcal{B}(I \times I))\) given \( \mathcal{G} \). But \( \mathcal{B}(I \times I) = \mathcal{B}(I) \otimes \mathcal{B}(I) \) and \( \lambda_2 = \lambda_1 \otimes \lambda_1 \) and

\[
\tilde{\nu}_{(x,y)}(A) := \lambda_1(A^x)
\]

is another such family for \((I \times I, \mathcal{B}(I \times I))\). As \( \mathcal{B}(I \times I) \) is countably generated, uniqueness forces

\[
\forall (x, y) \in J^* \ \forall A \in \mathcal{B}(I \times I): \quad \nu_{(x,y)}(A) = \lambda_1(A^x)
\]

on a set \( J^* \in \mathcal{B}(I \times I) \) with \( \lambda_2(J^*) = 1 \).

Pick \((x^*, y^*) \in J^* \) and \ldots
...note that \( \{x^*\} \times A \in \mathcal{B}(I \times I) \) for all \( A \in \mathcal{B}(I) \). Hence

\[
\forall A \in \mathcal{B}(I): \; \nu_{(x^*,y^*)}(\{x^*\} \times A) = \lambda_1(A)
\]

But, thanks to completeness of \( \lambda_2 \),

\[
\forall E \subseteq I: \; \{x^*\} \times E \in \mathcal{L}(I \times I)
\]

and so \( E \mapsto \nu_{(x^*,y^*)}(\{x^*\} \times E) \) is well-defined and \( \sigma \)-additive on \( 2^I \). In particular, \( \lambda_1 \) extends to all \( E \subseteq I \), a contradiction! \( \square \)
We now move to the main theorem:

Theorem

Let $\mu$ be a finite measure on standard Borel space $(X, \mathcal{B}(X))$ and let $\mathcal{G} \subseteq \mathcal{B}(X)$ be a $\sigma$-algebra. Then there exists a family $\{\nu_x : x \in X\}$ of conditional measures for $\mu$ given $\mathcal{G}$. Moreover, this family is unique $\mu$-a.e. and, if $\mathcal{G}$ is countably generated, then there is a version of $\{\nu_x : x \in X\}$ such that

$$\forall x \in X \ \forall A \in \mathcal{G} : \ \nu_x(A) = 1_A(x).$$

(We call such a version proper.)
Kuratowski’s theorem: \( \exists f: X \to \mathbb{R} \) bi-measurable injection.

Consider a version of \((1_{\{f \leq q\}})g\) for every \(q \in \mathbb{Q}\) and set

\[ F(q, x) := (1_{\{f \leq q\}})g(x) \]

As \(f \geq 0\), WLOG: \(F(q, x) := 0\) for \(q < 0\)

Lemma

\( \exists X^* \in \mathcal{G} \) with \(\mu(X \setminus X^*) = 0\) and

\( \forall x \in X^*: \ q \mapsto F(q, x) \) non-decreasing on \(\mathbb{Q}\)
Proof of a.e. monotonicity

Let

\[ X_{p,q} := \{ x \in X : F(p,x) \leq F(q,x) \} \]

Then \( X_{p,q} \in \mathcal{G} \) while

\[
\int_{\{F(p,\cdot)>F(q,\cdot)\}} [F(q,\cdot) - F(p,\cdot)] d\mu
\]

\[
= \int_{\{F(p,\cdot)>F(q,\cdot)\}} (1\{f\leq q\} - 1\{f\leq p\}) d\mu \geq 0
\]

shows \( \mu(X \setminus X_{p,q}) = 0 \). Definining

\[ X^* := \bigcap_{p,q \in \mathbb{Q}, p < q} X_{p,q} \]

we have \( X^* \in \mathcal{G} \) and \( \mu(X \setminus X^*) = 0 \).
Lemma

$\exists X^{**} \in G \text{ with } \mu(X \setminus X^{**}) = 0$ \text{ such that}

$\forall x \in X^{**}: \ q \mapsto F(q, x) \text{ right-continuous on } Q$
Proof of a.e. right continuity

Above argument: \( n \mapsto F(q + 1/n, \cdot) \) non-increasing \( \mu \)-a.e.

Dominated Convergence \( (F(q + 1/n, \cdot) \leq F(q + 1, \cdot) \) a.e.):

\[
\mu(f \leq q) = \lim_{n \to \infty} \mu(f \leq q + 1/n)
= \lim_{n \to \infty} \int F(q + 1/n, \cdot) \, d\mu
= \int \lim_{n \to \infty} F(q + 1/n, \cdot) \, d\mu \geq \int F(q, \cdot) \, d\mu = \mu(f \leq q),
\]

So

\[
\lim_{n \to \infty} F(q + 1/n, \cdot) = F(q, \cdot), \quad \mu\text{-a.e.}
\]

Set

\[
X^{**} := \bigcap_{q \in \mathbb{Q}} \{ x \in X^* : \limsup_{n \to \infty} F(q + 1/n, x) = F(q, x) \}
\]

Then \( X^{**} \in \mathcal{G} \) and \( \mu(X \setminus X^{**}) = 0. \)
Proof of existence

For \( x \in X^{**} \) and \( t \in \mathbb{R} \) set

\[
\tilde{F}(t, x) := \inf_{q \in Q \cap (t, \infty)} F(q, x)
\]

else \( \tilde{F}(t, x) := 1_{[0, \infty)}(t) \). Note: \( \tilde{F}(t, x) = 0 \) for \( t < 0 \).

As \( t \mapsto \tilde{F}(t, x) \) is non-decreasing, right-continuous for all \( x \in X \),
\( \exists \{\nu_x : x \in X\} \) Radon measures on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) such that

\[
\forall x \in X \forall t \in \mathbb{R} : \nu_x((-\infty, t]) = \tilde{F}(t, x)
\]

As \( \tilde{F}(q, x) = F(q, x) \) for \( q \in Q \) and \( x \in X^{**} \),

\[
\forall q \in Q : x \mapsto \nu_x((-\infty, q]) \text{ is a version of } (1_{\{f \leq q\}})_G
\]

We now proceed by an extension argument \ldots
The class
\[ \mathcal{P} := \{ \{ f \leq q \} : q \in \mathbb{Q} \} \]

is a \( \pi \)-system generating \( \mathcal{B}(X) \) (by bimeasurability of \( f \) and the fact that \( \{ (-\infty, q] : q \in \mathbb{Q} \} \) generates \( \mathcal{B}(\mathbb{R}) \)) while
\[ \mathcal{L} := \left\{ A \in \mathcal{B}(X) : x \mapsto \nu_x(A) \text{ is a version of } (1_A)_\mathcal{G} \right\} \]

is a \( \lambda \) system with \( \mathcal{P} \subseteq \mathcal{L} \). Hence \( \mathcal{B}(X) = \sigma(\mathcal{P}) \subseteq \mathcal{L} \).

Being a version of \( (1_A)_\mathcal{G} \) implies
\[ \forall A \in \mathcal{B}(X) \ \forall B \in \mathcal{G} : \mu(A \cap B) = \int_B \nu_x(A) d\mu \]

showing that \( \{ \nu_x : x \in X \} \) are conditional measures given \( \mathcal{G} \).
$\mathcal{F}$ countably generated ⇒ conditional measure unique a.e.

$\mathcal{G}$ countably generated: $\mathcal{G} = \sigma(\{A_n : n \geq 1\})$ and set $\mathcal{A} :=$ algebra generated by $\{A_n : n \geq 1\}$. We know

$$\forall A \in \mathcal{G}: (1_A)_\mathcal{G} = 1_A \mu\text{-a.e.}$$

so

$$X' := \bigcap_{A \in \mathcal{A}} \{x \in X : \nu_x(A) = 1_A(x)\}$$

obeys $X' \in \mathcal{G}$ and $\mu(X \setminus X') = 0$. As

$$\mathcal{L}' := \left\{ A \in \mathcal{B}(\mathbb{R}) : (\forall x \in X' : \nu_x(A) = 1_A(x)) \right\}$$

is a $\lambda$-system with $\mathcal{A} \subseteq \mathcal{L}'$ we have $\mathcal{G} = \sigma(\mathcal{A}) \subseteq \mathcal{L}'$. This shows that $\{\nu_x : x \in X\}$ is proper.