15. STANDARD BOREL SPACES

We will now delve deeper into the connection between topology and measurable structure. Specifically, we will identify a class of topological spaces that is relatively plain from the perspective of measure theory and yet rich enough to capture many examples of practical interest. We assume only minimal familiarity with basic concepts of topology and metric spaces; the needed facts will be recalled explicitly whenever relevant.

15.1 Polish and Standard Borel spaces.

We start by the basic definitions of this section:

Definition 15.1 (Polish space) A topological space is said to be Polish if the topology is metrizable and the resulting metric space — referred to as Polish metric space — is complete and separable.

Definition 15.2 (Standard Borel space) A standard Borel space is a measurable space \((X, \mathcal{F})\) such that \(X\) is a Polish space and \(\mathcal{F} = \mathcal{B}(X)\) is the \(\sigma\)-algebra of Borel sets induced by the Polish topology.

We remark that, while the same (Polish) topology can be metrized via several different (complete and separable) metrics, these will all induce the same class of Borel sets. Still, since a metric is always guaranteed to exists, we may henceforth restrict attention to Polish metric spaces and simply forget the metric whenever questions that only pertain to topology are of concern.

We start with some standard examples of Polish (and thus standard Borel) spaces. Recall that the discrete metric on a set \(X\) is defined by \(d(x, y) = 1\) if \(x = y\) and the discrete topology is then the collection of all subsets of \(X\).

Lemma 15.3 The following are Polish spaces:

1. \(\{1, \ldots, n\}\) with \(n \in \mathbb{N}\) or \(\mathbb{N}\) itself; equipped with discrete metric/topology.
2. \(\mathbb{R}\) endowed with the Euclidean-metric topology.
3. \(\mathbb{R}^n\), for \(n \geq 1\) integer, under Euclidean-metric topology.

Every non-empty closed subset of a Polish space, endowed with relative topology, is Polish.

Proof. Discrete topology is always complete because a sequence is Cauchy and/or convergent if and only if it is eventually constant. If \(X\) is countable, then the discrete-topology on \(X\) is also trivially separable. The reals \(\mathbb{R}\) are complete in the Euclidean metric by construction and so is any closed subset thereof. The rationals (which are countable) are then dense in \(\mathbb{R}\). The same argument applies to \(\mathbb{R}^n\) for all \(n \geq 1\).

Let \(X\) be a Polish space and let \(C \subseteq X\) be closed and non-empty. Then \(C\), if metrized by the Polish metric \(d\) on \(X\), is complete so we just need to show that it is separable. Let \(\{a_n\}_{n \geq 1}\) be a sequence enumerating a dense set in \(X\). For each \(n \geq 1\), let \(x_n \in C\) be a point with \(d(x_n, a_n) \leq 2 \inf \{d(x, a_n) : x \in C\}\). Now let \(y \in C\). Then for each \(\epsilon > 0\) there is \(n \geq 1\) with \(d(y, a_n) < \epsilon\). This implies \(d(y, x_n) < 3\epsilon\). Hence \(\{x_n\}_{n \geq 1}\) is dense in \(C\).

We note that other subsets of \(\mathbb{R}\) are Polish as well. For instance, \((0, 1)\) is Polish in the relative Euclidean topology, although the metric that realizes the Polish topology is not
the Euclidean metric. Rather, one has to use, e.g.,
\[ \rho(x, y) := |\tan(\pi(x - \frac{1}{2})) - \tan(\pi(y - \frac{1}{2}))| \]  
(15.1)

Since open sets are countable unions of disjoint open intervals, every non-empty open subset of \( \mathbb{R} \) (and in fact, every non-empty open subset of a Polish space, prove this!) is Polish as well. So are other sets, for instance \((0, 1]\). There are of course subsets of \( \mathbb{R} \) that are not Polish in the relative topology; e.g., the set of all rationals.

In order to discuss our next set of examples, recall that, given a family of topological spaces \( \{(X_{\alpha}, \mathcal{O}_{\alpha})\}_{\alpha \in I} \), the Cartesian product \( X := \times_{\alpha \in I} X_{\alpha} \) can be endowed with the product topology which is coarsest topology on \( X \) containing all the sets \( \times_{\alpha \in I} \mathcal{O}_{\alpha} \), where \( \mathcal{O}_{\alpha} \subseteq \mathcal{O}_{\alpha} \) for each \( \alpha \in I \), such that \( \{\alpha \in I : \mathcal{O}_{\alpha} \neq X_{\alpha}\} \) is finite. For products of a countable families \( \{(X_n, \mathcal{O}_n)\}_{n \geq 1} \) of metrizable topological spaces and the topology on \( \mathcal{O}_n \) metrized by metric \( \rho_n \), the product topology can be metrized by the product metric
\[ \rho(\{(x_{n})_{n \geq 1}, (y_{n})_{n \geq 1}\}) := \sum_{n \geq 1} 2^{-n} \frac{\rho_n(x_{n}, y_{n})}{1 + \rho_n(x_{n}, y_{n})}. \]  
(15.2)

With these in hand, we give:

**Lemma 15.4** If \( X \) is Polish then \( X^\mathbb{N} := \times_{i \geq 1} X \), endowed with the product topology, is Polish. In particular, the following spaces are Polish:

1. \( \mathbb{R}^\mathbb{N} \) with product Euclidean topology.
2. Cantor space \( \{0, 1\}^\mathbb{N} \) with product discrete topology.
3. Baire space \( \mathbb{N}^\mathbb{N} \) with product discrete topology.
4. Hilbert cube \( [0, 1]^\mathbb{N} \) with product Euclidean topology.

**Proof.** The topology in \( X^\mathbb{N} \) is metrized by a product metric of the form (15.2). A sequence in \( X^\mathbb{N} \) is then Cauchy/convergent if and only if each of its coordinates is Cauchy/convergent. Since \( X \) is complete, \( X^\mathbb{N} \) is complete. For separability let \( A \subseteq X \) be a countable dense subset and pick \( x_0 \in A \). Then let \( A' \) be the set of sequences \( \{x_i\}_{i \geq 1} \subseteq A^\mathbb{N} \) such that \( \{i \geq 1 : x_i \neq x_0\} \) is finite. Then \( A' \) is countable and, since we rely on product topology, dense in \( X^\mathbb{N} \). The second part of the claim follows from Lemma 15.3. \( \square \)

Many Polish spaces of interest arise in somewhat more complicated considerations than those used above, e.g., in probability or functional analysis. We leave to the reader to verify that the following are also Polish spaces:

1. \( L^p(X, \mathcal{F}, \mu) \) with \( 1 \leq p < \infty \) where \( \mathcal{F} \) is countably generated and \( \mu \) is \( \sigma \)-finite. (This includes the \( \ell^p(\mathbb{N}) \) spaces with \( p \geq 1 \) finite.)
2. \( \ell^\infty(\mathbb{N}) \) fails to be Polish (because, while complete, it is not separable) but its subspace of converging sequences,
\[ c_0 := \{ \{x_{n}\}_{n \geq 1} \in \ell^\infty(\mathbb{N}) : x_{n} \rightarrow 0 \}, \]  
(15.3)

endowed with the supremum metric is Polish.
3. The set \( C(X) \) of all continuous functions on a compact metric space \( X \) is Polish in the metric and topology induced by the supremum norm.
(11) The vector space $L(X, Y)$ of bounded linear operators $T: X \to Y$ where $X$ and $Y$ are Banach spaces with $X$ separable is Polish in the strong topology. This topology is defined as the coarsest topology containing the neighborhood

$$\{ T' \in L(X, Y) : \| T'x - Tx \|_Y < \epsilon \}$$

(15.4)

for all $T \in L(X, Y)$, all $x \in X$ and all $\epsilon > 0$. The associated notion of strong convergence of a sequence of operators means

$$T_n \overset{\text{strong}}{\to} T : \forall x \in X : T_n x \to Tx \text{ in } Y\text{-norm.}$$

(15.5)

With $X$ separable, the strong topology is metrized, e.g., by

$$d(T_1, T_2) := \sum_{n \geq 1} 2^{-n} \max\{ \| T_1 x_n - T_2 x_n \|_Y, 1 \},$$

(15.6)

where $\{ x_n \}_{n \geq 1}$ is a countable dense set in $X$. We note that, in infinitely dimensional spaces, the norm topology on $L(X, Y)$ fails to be Polish.

(12) The set $M_1(X, \mathcal{F})$ of all probability measures on a standard Borel space $(X, \mathcal{F})$ is Polish in the topology of so called weak convergence. This topology is the coarsest topology on $X$ containing the neighborhood

$$\left\{ \nu \in M_1(X, \mathcal{F}) : \left| \int f \, d\nu - \int f \, d\mu \right| < \epsilon \right\}$$

(15.7)

for all $\mu \in M_1(X, \mathcal{F})$, all $f \in \mathcal{C}_b(X)$ and $\epsilon > 0$. The weak convergence thus corresponds to

$$\mu_n \overset{\text{weak}}{\to} \mu := \forall f \in \mathcal{C}_b(X) : \int f \, d\mu_n \to \int f \, d\mu.$$  

(15.8)

The above topology is metrized by

$$d(\mu, \nu) := \sum_{n \geq 1} 2^{-n} \max\left\{ \left| \int f_n \, d\mu - \int f_n \, d\nu \right|, 1 \right\},$$

(15.9)

where $\{ f_n \}_{n \geq 1}$ is a countable dense set in $\mathcal{C}_b(X)$ which is separable thanks to $X$ being itself separable.

Note that the last example leads to a hierarchy of standard Borel spaces: probability measures on $(X, \mathcal{F})$, probability measures on probability measures on $(X, \mathcal{F})$, etc.

Our aim to to study bijections between Polish spaces that preserve the underlying measure structure. Open continuous bijections, a.k.a. homeomorphisms, are examples of these, but they are tied too closely to the topology and thus preserve too much structure of the underlying spaces. (For instance, they preserve connectivity and compactness.) Better adapted to our needs are maps that fall under:

**Definition 15.5** (Borel isomorphism) A map $f: X \to Y$ between topological spaces $X, Y$ is called **Borel isomorphism** if $f$ is bijective and both $f$ and $f^{-1}$ are Borel measurable.
While continuous maps preimage one topology into the other, Borel measurable maps only need to preimage the topology into the class of Borel sets. As a consequence, although there are many non-homeomorphic Polish spaces, there are only three basic examples of standard Borel spaces:

**Theorem 15.6 (Kuratowski)** Let \((X, \mathcal{B}(X))\) be a standard Borel space. Then there is a Borel isomorphism \(f : (X, \mathcal{B}(X)) \rightarrow (Y, \mathcal{B}(Y))\) onto a Polish space \(Y\) where

\[
Y = \begin{cases} 
\{1, \ldots, n\}, & \text{if } |X| = n \text{ for some } n \in \mathbb{N}, \\
\mathbb{N}, & \text{if } X \text{ is countably infinite}, \\
[0, 1], & \text{otherwise}.
\end{cases}
\] (15.10)

The rest of this section will be spent on the proof of Kuratowski’s theorem. Before we delve into the proof, we note its simple consequence:

**Corollary 15.7** Every infinite Polish space is either countable or of cardinality of the continuum. In particular, the Continuum Hypothesis holds in the class of Polish spaces.

**Proof.** This is a direct consequence of two bottom lines in (15.10). \(\square\)

We will actually prove this corollary directly as part of the proof of Kuratowski’s theorem; see Proposition 15.15.

### 15.2 Schröder-Bernstein’s theorem.

The proof of Theorem 15.6 hinges on some standard arguments from descriptive set theory and point set topology. First we consider a slight extension of a classical theorem that underpins the definition of cardinality:

**Theorem 15.8 (Schröder-Bernstein)** Let \(A \subseteq X\) and \(B \subseteq Y\) be sets such that there exist bijections \(f : X \rightarrow B\) and \(g : Y \rightarrow A\). Then there exists a bijection \(h : X \rightarrow Y\). Moreover, given \(\sigma\)-algebras \(\mathcal{F} \subseteq 2^X\) and \(\mathcal{G} \subseteq 2^Y\), if \(A \in \mathcal{F}\) and \(B \in \mathcal{G}\) and if \(f, g^{-1}\) are \(\mathcal{F}/\mathcal{G}\)-measurable and \(f^{-1}, g\) are \(\mathcal{G}/\mathcal{F}\)-measurable, then \(h\) can be taken \(\mathcal{F}/\mathcal{G}\)-measurable.

**Proof.** Note that the map \(g \circ f\) images \(X\) injectively into \(A\). Denote by \(C := (g \circ f)(X)\) the range of this map. For each \(x \in X\) let

\[
n(x) := \sup\{n \geq 0 : x \in \text{Ran}((g \circ f)^n)\}
\] (15.11)

where \((g \circ f)^0(x) := x\). Decomposing \(X\) according to whether \(n(x) < \infty\) or \(n(x) = \infty\) and, in the former case, whether \((g \circ f)^{-n(x)}(x)\) belongs to \(X \setminus A\) or \(A \setminus C\) (noting that \((g \circ f)^{-n(x)}(x)\) cannot belong to \(C\) because then at least one more \((g \circ f)^{-1}\) could be applied to it) shows that the sets

\[
\{(g \circ f)^n(X \setminus A)\}_{n \geq 0} \quad \{(g \circ f)^n(A \setminus C)\}_{n \geq 0} \quad \bigcap_{n \geq 0} (g \circ f)^n(X)
\] (15.12)
form a (disjoint) partition of \(X\). We define
\[
h(x) := \begin{cases} 
  f(x), & \text{if } x \in \bigcup_{n \geq 0} (g \circ f)^n(X \setminus A), \\
  g^{-1}(x), & \text{if } x \in \bigcup_{n \geq 0} (g \circ f)^n(A \setminus C), \\
  g^{-1}(x), & \text{if } x \in \bigcap_{n \geq 0} (g \circ f)^n(X).
\end{cases}
\] (15.13)

Next we claim that \(g \circ h : X \to A\) is a bijection. Indeed, in the latter two cases in (15.13) the map \(g \circ h\) is the identity, while in the first case \(g \circ h\) maps \((g \circ f)^n(A \setminus C)\) bijectively onto \((g \circ f)^{n+1}(A \setminus C)\). Hence \(g \circ h\) maps \(X\) bijectively onto
\[
(\bigcup_{n \geq 1} (g \circ f)^n(X \setminus A)) \cup (\bigcup_{n \geq 0} (g \circ f)^n(A \setminus C)) \cup \bigcap_{n \geq 0} (g \circ f)^n(X) = A.
\] (15.14)

It follows that \(h\) maps \(X\) bijectively onto \(g^{-1}(A) = Y\).

If \(A \in \mathcal{F}\) and \(B \in \mathcal{G}\), and both \(f, g^{-1}\) and \(g, f^{-1}\) are measurable as stated, the map \(g \circ f\) is \(\mathcal{F} / \mathcal{G}\)-measurable and so \(C = (g \circ f)(X)\), and thus all sets in (15.12) belong to \(\mathcal{F}\). It follows that \(h\) is \(\mathcal{F} / \mathcal{G}\)-measurable as desired. \(\square\)

Theorem 15.8 tells us that, in order to link two standard Borel spaces via a bimeasurable bijection, it suffices demonstrate the existence of a bimeasurable embedding of one space into the other and vice versa. To illustrate this approach, we will now use it to construct Borel isomorphisms between the Cantor space and the Hilbert cube, which are the spaces that will play a prominent role in the proof of Theorem 15.6.

**Proposition 15.9** The standard Borel spaces \(\mathcal{C} := \{0, 1\}^\mathbb{N}\) and \(\mathcal{H} := [0, 1]^\mathbb{N}\), equipped with measure structure arising from their natural Polish topology, are Borel isomorphic.

We start by showing:

**Lemma 15.10** \(\mathcal{C}\) is Borel isomorphic to \([0, 1]\).

**Proof.** Consider the map \(f : \mathcal{C} \to [0, 1]\) defined, for \(\sigma = \{\sigma_n\}_{n \geq 1} \in \mathcal{C}\), by
\[
f(\sigma) := \sum_{n \geq 1} \frac{\sigma_n}{3^n}.
\] (15.15)

Let us use \(\varrho(\sigma, \sigma') := \sum_{n \geq 1} 3^{-n} |\sigma_n - \sigma'_n|\) to metrize the product discrete topology on \(\mathcal{C}\). Given any \(\sigma \neq \sigma'\), denote \(n := \inf\{i \geq 1 : \sigma_i \neq \sigma'_i\}\). Then
\[
|f(\sigma) - f(\sigma')| - 3^{-n} \leq \sum_{k > n} 3^{-n} = \frac{1}{2} 3^{-n}.
\] (15.16)

while
\[
|\varrho(\sigma, \sigma') - 3^{-n}| \leq \sum_{k > n} 3^{-n} = \frac{1}{2} 3^{-n}.
\] (15.17)

It follows that
\[
3 \varrho(\sigma, \sigma') \geq |f(\sigma) - f(\sigma')| \geq \frac{1}{3} \varrho(\sigma, \sigma')
\] (15.18)

thus proving that \(f\) is injective, continuous and open. Since \(A := \text{Ran}(f)\) is closed, \(f\) is bi-measurable with a measurable image.
Next we will construct a map \( g: [0,1] \to \mathcal{C} \). Given \( x \in [0,1) \), set
\[
\forall i \geq 1: \ \sigma_i := \lfloor 2^i x \rfloor \mod 2. \tag{15.19}
\]
Then \( \sigma_i \in \{0,1\} \) and so we can let \( g(x) := \{\sigma_i\}_{i \geq 1} \). In order to include \( x = 1 \), set \( g(1) := (1,1, \ldots) \). Then \( x \) can be computed from \( \{\sigma_i\}_{i \geq 1} \) via
\[
x = \sum_{i \geq 1} \frac{\sigma_i}{2^i}, \tag{15.20}
\]
and so \( g \) is invertible on its range which is \( \mathcal{C} \) minus the sequences that end with an infinite progression of 1’s, except for the sequence \((1,1, \ldots)\). Since the set of excluded sequences is countable, we have \( \text{Ran}(g) \in \mathcal{B}([0,1]) \). The formula (15.20) shows that \( g^{-1} \) is continuous and thus measurable. That \( g \) is itself measurable follows from
\[
g^{-1}\left( \{\sigma_i\}_{i \geq 1} : (\forall i \leq n: \sigma_i = \hat{\sigma}_i) \right) = \sum_{i=1}^{n} \frac{\hat{\sigma}_i}{2^i} + [0,2^n), \tag{15.21}
\]
and the fact that the sets \( \{\sigma_i\}_{i \geq 1} : (\forall i \leq n: \sigma_i = \hat{\sigma}_i) \), with \( n \geq 1 \) and \( \hat{\sigma} \in \mathcal{C} \) generate the topology in \( \mathcal{C} \) and thus also the Borel sets \( \mathcal{B}(\mathcal{C}) \). Hence, also \( g \) is bi-measurable with measurable image. By Theorem 15.8, \( f \) and \( g \) give rise to a Borel isomorphism of \(([0,1], \mathcal{B}([0,1])) \) and \((\mathcal{C}, \mathcal{B}(\mathcal{C})) \).

Next we observe:

**Lemma 15.11** Let \( X \) and \( Y \) be Polish spaces and let \( h : X \to Y \) be a Borel isomorphism. Then
\[
h^{N}(\{x_i\}_{i \geq 1}) := \{h(x_i)\}_{i \geq 1} \tag{15.22}
\]
is a Borel isomorphism \( h^{N}: X^{N} \to Y^{N} \).

**Proof.** It is clear that \( h^{N} \) is a bijection of \( X^{N} \) onto \( Y^{N} \) so we just need to prove that it is bimeasurable. Let \( \mathcal{O} \) be the topology in \( X \). The topology in \( X^{N} \), and thus \( \mathcal{B}(X^{N}) \), is then generated by sets \( \times_{i \geq 1} O_i \) where \( \{O_i\}_{i \geq 1} \subseteq \mathcal{O} \) is such that \( \{i \geq 1 : O_i \neq X\} \) is finite. It follows that \( \mathcal{B}(X^{N}) = \mathcal{\sigma}(\mathcal{S}_X) \) where
\[
\mathcal{S}_X := \left\{ \times_{i \geq 1} A_i : \{A_i\}_{i \geq 1} \subseteq \mathcal{B}(X) \wedge \{i \geq 1 : A_i \neq X\} \text{ is finite} \right\}. \tag{15.23}
\]
As is readily checked, \( \mathcal{S}_X \) is a semialgebra and so is the corresponding object \( \mathcal{S}_Y \) for \( Y \). Since \( h^{N} \) images \( \mathcal{S}_X \) onto \( \mathcal{S}_Y \) bijectively, Dynkin’s \( \pi/\lambda \)-Theorem ensures that \( h^{N} \) images \( \mathcal{B}(X) := \mathcal{\sigma}(\mathcal{S}_X) \) bijectively onto \( \mathcal{B}(Y) := \mathcal{\sigma}(\mathcal{S}_Y) \). Hence \( h^{N} \) and its inverse are Borel measurable and so \( h^{N} \) is a Borel isomorphism.

We are now ready to give:

**Proof of Proposition 15.9.** By Lemma 15.10, \( \mathcal{C} \) is Borel isomorphic to \([0,1] \) and so, by Lemma 15.11, \( \mathcal{C}^{N} \) is Borel isomorphic to \([0,1]^{N} \). Since \( \mathbb{N} \) is in a bijective correspondence with \( \mathbb{N} \times \mathbb{N} \), the space \( \mathcal{C}^{N} = \{0,1\}^{N \times \mathbb{N}} \) is Borel isomorphic with \( \{0,1\}^{N} = \mathcal{C} \). This identifies \([0,1], \mathcal{C} \) and \( \mathcal{M} \) by Borel isomorphisms. □
15.3 Proof of Kuratowski’s theorem.

We will now move to the proof of Theorem 15.6. Relying on Proposition 15.9, this will be done by constructing a bimeasurable embedding of any Polish space into the Hilbert cube (Proposition 15.13) and the Cantor space into any Polish space (Proposition 15.15). These embeddings will arise from open and continuous injections; i.e., homeomorphisms onto a subset endowed with the relative topology. In order to verify the measurability assumptions of Theorem 15.8 we thus need to characterize the image of a Polish space under a homeomorphism. Recall that $G_δ$ is the class of sets in a topological space that are countable intersections of open sets. We then have:

**Lemma 15.12.** Let $X$ and $Y$ be Polish and let $g: X \to B$ be a homeomorphism (i.e., a bijection that is open and continuous) onto $B \subseteq Y$. Then $B$ is a $G_δ$ subset of $Y$.

**Proof.** Denote $f := g^{-1}$. Then $f$ is a homeomorphism of $B$ onto $X$. Using the metric $q$ that is bounded and makes $X$ a Polish metric space, define the oscillation of $f$ at $y \in \overline{B}$ by

$$\text{osc}_f(y) := \inf \left\{ \text{diam}_X(f(U)) : y \in U \land U \subseteq Y \text{ open} \right\}. \quad (15.24)$$

(This is well defined because every neighborhood of $y \in \overline{B}$ contains a point in $B$ on which $f$ is defined.) We claim

$$B = \overline{B} \cap \{ y \in \overline{B} : \text{osc}_f(y) = 0 \}. \quad (15.25)$$

Indeed, by continuity of $f$ we have $\text{osc}_f(y) = 0$ for all $y \in B$ and so all we have to prove is $\partial B \cap \{ \text{osc}_f = 0 \} \subseteq B$. Suppose $y \in \partial B$ is such that $\text{osc}_f(y) = 0$. Since points of $\partial B$ are accessible from $B$, there exists $\{ y_n \}_{n \geq 1} \subseteq B$ with $y_n \to y$. Now

$$\text{osc}_f(y) = 0 \Rightarrow \{ f(y_n) \}_{n \geq 1} \text{ is Cauchy} \quad (15.26)$$

because if $\varepsilon := \lim_{n \to \infty} \sup_{k,m > n} q(f(y_k), f(y_m)) > 0$, then for every $U \subseteq Y$ open with $y \in U$, we would have $\text{diam}_X(Y) \geq \varepsilon$. By completeness of $X$, the limit $x := \lim_{n \to \infty} f(y_n)$ exists and the continuity of $f^{-1} = g$ ensures that $y_n = g(f(y_n)) \to g(x)$ as $n \to \infty$. Hence, $y \in B$ as desired.

The function $\text{osc}_f$ is upper-semicontinuous (meaning that $\{ \text{osc}_f < a \}$ is open for each $a \in \mathbb{R}$). Since

$$\{ \text{osc}_f = 0 \} = \bigcap_{n \geq 1} \{ \text{osc}_f < 1/n \}, \quad (15.27)$$

the set $\{ \text{osc}_f = 0 \}$ is $G_δ$. As closed sets are $G_δ$ in metric spaces, by (15.25) so is $B$. \qed

We now state and prove:

**Proposition 15.13 (Universality of Hilbert cube)** Every Polish space is homeomorphic to a $G_δ$-subset of $\mathcal{H} = [0, 1]^N$.

**Proof.** Let $(X, q)$ be a Polish metric space with $q \leq 1$ (otherwise map $(X, q)$ bijectively onto $(X, \frac{q}{1+q})$ using the identity map). Let $\{x_n\}_{n \geq 1}$ be a sequence enumerating a dense subset of $X$. Define $f: X \to [0, 1]^N$ by

$$x \in X \mapsto f(x) := \{ q(x, x_n) \}_{n \geq 1} \in [0, 1]^N. \quad (15.28)$$
Then \( f \) is continuous and one-to-one and so there exists \( f^{-1}: f(X) \to X \). We claim that \( f^{-1} \) is also continuous. Indeed, let \( \{z_n\}_{n \geq 1} \subseteq X \) be a sequence such that \( f(z_n) \to y \in f(X) \). Let \( z \in X \) be such that \( y = f(z) \). We want to show that \( z_n \to z \). Given \( \epsilon > 0 \), find \( n \geq 1 \) such that \( \varrho(x_n, z) < \epsilon \). Then

\[
\varrho(z_k, z) \leq \varrho(z_k, x_n) + \varrho(z, x_n)
= f(z_k)_n + \varrho(z, x_n) \quad \xrightarrow{k \to \infty} \quad f(z)_n + \varrho(z, x_n) = 2\varrho(z, x_n) < 2\epsilon. \quad (15.29)
\]

As this holds for all \( \epsilon > 0 \), it follows that \( z_k \to z \) or, in explicit terms, \( f^{-1}(f(z_k)) \to f^{-1}(f(z)) \) as \( k \to \infty \). So \( f^{-1} \) is continuous on \( f(X) \). Thus \( f: X \to f(X) \) is a homeomorphism and so, by Lemma 15.12, \( f(X) \) is a \( G_\delta \) subset of \( [0, 1]^\mathbb{N} \).

The previous proposition gives the desired embedding of \( X \) into the Hilbert cube and, by Proposition 15.9, a Borel isomorphism into the Cantor space. It remains to construct an embedding of the Cantor space into a general (uncountable) Polish space. Here we will need another classical result from basic point-set topology:

**Theorem 15.14 (Cantor-Bendixon)** Let \( X \) be a second-countable topological space (i.e., such that there is a countable collection of open sets generating the topology by unions). Then there exist unique \( P, O \subseteq X \) such that

1. \( P \) is perfect (i.e., a closed set with no isolated points),
2. \( O \) is countable open,

and such that

\[
P \cap O = \emptyset \quad \land \quad X = P \cup O. \quad (15.30)
\]

**Proof of Theorem 15.14, existence.** Denote by \( T \) the topology on \( X \) and let \( \{U_n\}_{n \geq 1} \) be a sequence enumerating a (countable) basis of \( T \). Denote

\[
O := \bigcup \{U \in T : U \text{ countable}\} = \bigcup \{U_n : U_n \text{ countable}\}, \quad (15.31)
\]

where the equality follows from the fact that each \( U \in T \) is the union of a subsequence from \( \{U_n\}_{n \geq 1} \). Then \( O \) is open and countable while \( P := X \setminus O \) is closed. Let \( x \in P \). If \( U \) is open with \( x \in U \), then \( U \) — being the union of a subsequence from \( \{U_n\}_{n \geq 1} \) — is necessarily uncountable, because otherwise \( x \) would end up in \( O \). This means that \( P \cap U \setminus \{x\} \neq \emptyset \) for each \( U \) open with \( x \in U \). Hence, no \( x \in P \) is isolated. This shows that a pair \( (P, O) \) of a perfect and countable-open set satisfying \( (15.30) \) exists. \( \square \)

We postpone the proof of uniqueness to the end of the section. With the existence part of Theorem 15.14 in hand, we state:

**Proposition 15.15** (Cantor space to Polish space) Every uncountable Polish space contains a homeomorphic copy of \( \mathcal{C} \).

**Proof.** Let \( X \) be an uncountable Polish space. Every separable metric space is second countable, so let \( X = P \cup O \) be its decomposition into a disjoint pair consisting of a perfect set and a countable open set. Then \( P \), being nonempty (because \( X \) is uncountable) and closed, is Polish too and, since we are only looking for embeddings of \( \mathcal{C} \), we may as well assume that \( X = P \); i.e., that \( X \) contains no isolated points.
We will now present a construction that identifies a copy of \( \mathcal{C} \) in \( X \). Let \( \{x_i\}_{i \geq 1} \) be a sequence (of distinct points) enumerating a countable dense subset of \( X \). Denoting by \( q \) a Polish metric on \( X \), write \( B(x, r) := \{ y \in X : q(x, y) < r \} \) for the open ball of radius \( r \) centered at \( x \). We claim that

\[
i(n) := \inf\{ i > n : B(x_i, 2^{-i}) \subseteq B(x_n, 2^{-n}) \setminus B(x_n, 2^{-i}) \}\]

and

\[
k(n) := \inf\{ k > i(n) : B(x_k, 2^{-k}) \subseteq B(x_n, 2^{-i(n)}) \}\]

obey \( i(n) < \infty \) and \( k(n) < \infty \) for all \( n \geq 1 \). For this note that, since \( x_n \) is not isolated, \( \{x_j \in B(x_n, 2^{-n-1}) : j \geq 1 \} \) is infinite. No points of this set are isolated either and so there must be \( x_j \) with \( 2^{-i+1} < q(x_i, x_n) < 2^{-n-1} \). This proves \( i(n) < \infty \). The fact that \( x_n \) is not isolated even in \( B(x_n, 2^{-i(n)-1}) \) then shows \( k(n) < \infty \).

Next we will define a collection of open balls

\[
\left\{ B_{\sigma} : \sigma \in \bigcup_{n \geq 0} \{0,1\}^n \right\}
\]

centered at the points of \( \{x_i\}_{i \geq 1} \) recursively as follows: For \( n := 0 \), set

\[
B_{\nu} := B(x_1, 1/2).
\]

Assuming that, for some \( n \geq 0 \) and some \( \sigma \in \{0,1\}^n \), the ball \( B_{\sigma} \) has been defined, we denote by \( m \) the natural such that \( B_{\sigma} \) is centered at \( x_m \). Then we set

\[
B_{\sigma 0} := B(x_{(m)}, 2^{-i(m)}) \wedge B_{\sigma 1} := B(x_{k(m)}, 2^{-k(m)}),
\]

where \( \sigma \eta \), for \( \sigma = (\sigma_1, \ldots, \sigma_n) \in \{0,1\}^n \) and \( \eta \in \{0,1\} \), denotes the concatenated string \( (\sigma_1, \ldots, \sigma_n, \eta) \in \{0,1\}^{n+1} \). The construction ensures that

\[
\forall \sigma \in \bigcup_{n \geq 0} \{0,1\}^n : \quad B_{\sigma 0}, B_{\sigma 1} \subseteq B_{\sigma} \wedge B_{\sigma 0} \cap B_{\sigma 1} = \emptyset.
\]

For each \( n \geq 0 \), let

\[
r_n := \text{minimum of radii of the balls } \{ B_{\sigma} : \sigma \in \{0,1\}^n \}.
\]

Note that \( 0 < r_n \leq 2^{-n} \) for all \( n \geq 0 \).

Consider an infinite string \( \sigma = (\sigma_1, \sigma_2, \ldots) \in \{0,1\}^\mathbb{N} \). The completeness of \( X \) ensures that the intersection \( \bigcap_{n \geq 1} B(\sigma_1, \ldots, \sigma_n) \) is non-empty and contains exactly one point \( y_{\sigma} \); namely, the limit of the sequence of the centers of these balls. Consider the map \( \mathcal{C} \to X \) defined by \( \sigma \mapsto y_{\sigma} \). If \( \sigma \) and \( \tilde{\sigma} \) agree in the first \( n \) digits and \( \sigma_{n+1} = 0 \) and \( \tilde{\sigma}_{n+1} = 1 \), then \( y_{\sigma} \in B(\sigma_1, \ldots, \sigma_n, 0) \) and \( y_{\tilde{\sigma}} \in B(\sigma_1, \ldots, \sigma_n, 1) \) and so, by (15.37),

\[
r_n \leq q(y_{\sigma}, y_{\tilde{\sigma}}) \leq 2^{-n+1}.
\]

This shows that \( \sigma \mapsto y_{\sigma} \) is one-to-one and continuous. To show that the inverse map is continuous, let us note that if \( q(y_{\sigma}, y_{\tilde{\sigma}}) < r_n \), then \( \sigma_i = \tilde{\sigma}_i \) for \( i = 1, \ldots, n \). As \( r_n \to 0 \), the smaller \( q(y_{\sigma}, y_{\tilde{\sigma}}) \), the larger part of \( \sigma \) and \( \tilde{\sigma} \) must be the same. Hence, the inverse map is continuous and \( \sigma \mapsto y_{\sigma} \) is thus a homeomorphic embedding \( \mathcal{C} \to X \).

The argument from the proof of Proposition 15.15 shows:
Corollary 15.16  If $S \subseteq X$ is perfect and $U \subseteq X$ open in a Polish space $X$, then
\[ S \cap U \neq \emptyset \implies S \cap U \text{ is uncountable.} \quad (15.40) \]
In particular, if $U$ is also countable, then $S \cap U = \emptyset$.

Proof. If $S \cap U$ is non-empty, then $S \cap U$ contains an open ball in the relative topology on $S$. Since $S$ is Polish in the relative topology, that ball in turn embeds a Cantor space, which is uncountable.

With this we now complete:

Proof of Theorem 15.14, uniqueness. Let $X$ be a Polish space and $P', O'$ be subsets of $X$ such that $P'$ is perfect and $O$ countable-open set with $X = P' \cup O'$. As $O$ is countable, (15.40) forces $P' \cap O = \emptyset$, i.e., $P' \subseteq P$. On the other hand, $O'$ is countable open and so it appears in the union (15.31) defining $O$, thus forcing $O' \subseteq O$. As $P' \cup O' = X$, we must have $P' = P$ and $O' = O$ as desired.

Having arrived to the finale, we are finally ready to put all pieces together and prove Kuratowski’s Theorem:

Proof of Theorem 15.6. If $X$ is finite or countable, then the topology on $X$ is discrete and the desired Borel isomorphism is constructed directly. So assume that $X$ is uncountable. Proposition 15.15 gives a homeomorphic embedding $\mathcal{C} \hookrightarrow X$ while Proposition 15.13 gives a homeomorphic embedding $X \hookrightarrow \mathcal{H}$. Both of these image into a $\mathcal{G}_d$-set by Lemma 15.12 and so the inverse maps are Borel measurable. Moreover, by Proposition 15.9, the spaces $\mathcal{C}$ and $\mathcal{H}$ are Borel isomorphic. Composing the map $X \hookrightarrow \mathcal{H}$ with the isomorphism between $\mathcal{C}$ and $\mathcal{H}$, we get a bimesurable embedding $X \hookrightarrow \mathcal{C}$. Theorem 15.8 then shows that $X$ is Borel isomorphic to $\mathcal{C}$ (and to $\mathcal{H}$ as well, of course).

The term standard Borel space has been introduced by G.W. Mackay in “Borel structure in groups and their duals” (Trans. Am. Math. Soc., 85 (1957) 134–165) although the subject goes back to the Polish school of mathematics in the 1920s and 1930s. (Hence the name Polish space.) We will appreciate the structural benefits of standard Borel spaces when we work with measures on finite or infinite product spaces.