Problem 1: Let \( f \in C^1(\mathbb{R}) \) have compact support and let \( Hf(x) := \frac{1}{\pi} \int \frac{f(x-t)-f(x+t)}{t} \, dt \) be its Hilbert transform. Prove
\[
\lim_{x \to \pm \infty} xHf(x) = \frac{1}{\pi} \int f \, d\lambda
\]
Conclude that, if \( \int f \, d\lambda \neq 0 \), then \( Hf \notin L^1 \).

Problem 2: Let \( p \in (1, \infty] \) and suppose \( \mu \) is a \( \sigma \)-finite measure on \( (X, \mathcal{F}) \). Prove that there are constants \( c_1(p), c_2(p) \in (0, \infty) \) such that
\[
c_1[f]_p \leq \inf \left\{ B > 0 : A \in \mathcal{F}, \mu(A) < \infty \Rightarrow \int_A |f| \, d\mu \leq B\mu(A)^{1-1/p} \right\} \leq c_2[f]_p
\]
holds for all \( f \in L^{p,\infty}(X, \mathcal{F}, \mu) \). (Note that this fails for \( p = 1 \), because the finiteness of the infimum would imply \( f \in L^1 \), but works fine for \( p = \infty \).) In addition, prove that
\[
\|f\|_{p,\infty} := \sup_{A \in \mathcal{F}, \mu(A) < \infty} \frac{1}{\mu(A)^{1-1/p}} \int_A |f| \, d\mu
\]
is a (proper) norm on \( L^{p,\infty} \).

Problem 3: Prove that if \( T \) is sublinear and restricted weak type \( (p_0, q_0) \) and \( (p_1, q_1) \), then it is restricted weak type \( (p_\theta, q_\theta) \) for all \( \theta \in [0, 1] \), where
\[
(p_\theta^{-1}, q_\theta^{-1}) = \theta(p_1^{-1}, q_1^{-1}) + (1 - \theta)(p_0^{-1}, q_0^{-1})
\]
is the usual interpolation of the indices.

Problem 4: Let \( \alpha \in (0, 1) \) and and consider the Riemann-Liouville fractional integral
\[
I_\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) \, dt,
\]
on measurable functions \( f : [0, \infty) \to \mathbb{R} \). Prove that \( I_\alpha \) is weak type \( (1, \frac{1}{1-\alpha}) \) and strong type \( (p, q) \) for all \( p \in (1, \alpha^{-1}) \) and \( q \) defined by \( q^{-1} = p^{-1} - \alpha \).

Problem 5: Show that if the Marcinkiewicz Theorem were true without the restriction \( p_i \leq q_i \), then \( f \in L^4(\mathbb{R}^d) \) and \( g \in L^{4,w}(\mathbb{R}^d) \) would imply \( fg \in L^2(\mathbb{R}^d) \). Then find functions \( f, g \) for which this conclusion fails — and thus prove that the requirements \( p_i \leq q_i \) in the Marcinkiewicz Theorem are essential.

Problem 6: Consider functions on measure space \( (\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d), \lambda) \). For \( f \in L^1 \), define the partial Fourier inverse by
\[
T_n f(x) := \int_{-n}^{n} \hat{f}(k)e^{-2\pi i k \cdot x} \, dk,
\]
where \( \hat{f} \) is the Fourier transform of \( f \). Let \( p \in (1, \infty) \). Prove that for each \( n \geq 1 \), the operator \( T_n \) extends continuously to a map \( L^p \to L^p \) and

\[
\forall f \in L^p : \quad T_n f \underset{n \to \infty}{\longrightarrow} f \quad \text{in} \quad L^p.
\]

**Problem 7:** Let \( \mu \) be a finite Borel measure on \( \mathbb{R} \) and define its Fourier transform by

\[
\hat{\mu}(k) := \int e^{2\pi ikx} \mu(dx)
\]

Prove the following:

1. \( \hat{\mu} \) is a bounded, continuous function.
2. If \( \nu \) is another finite Borel measure such that \( \hat{\nu} = \hat{\mu} \) then \( \nu = \mu \).
3. If \( \hat{\mu} \in L^1 \), then \( \mu = f \, d\lambda \) where \( f := (\hat{\mu})^\vee \).
4. For \( p \in [1, 2] \) and \( q \) such that \( p^{-1} + q^{-1} = 1 \), the Hausdorff-Young inequality

\[
\|\hat{\mu}\|_q \leq \|\mu\|_p := \sup \left\{ \int g d\mu : g \in C_c(\mathbb{R}) \land \|g\|_q \leq 1 \right\}
\]

holds, where \( \| \cdot \|_q \) (on both sides) is with respect to the Lebesgue measure.