Problem 1: (Conditions for nesting of $L^p$ spaces) Suppose $(X, F, \mu)$ is a measure space and let $p, r \in (0, \infty)$ obey $p < r$. Then

$$L' \subseteq L^p \iff \exists c > 0: \mu(F) \cap (c, \infty) = \emptyset$$

and

$$L^p \subseteq L' \iff \exists c > 0: \mu(F) \cap (0, c) = \emptyset$$

Problem 2: (Limits of $p$-norm as $p \to \infty$ or $p \to 0$) Given a measure space $(X, F, \mu)$ with $\mu$ finite, prove that

$$\forall f \in L^0: \|f\|_\infty = \lim_{p \to \infty} \|f\|_p$$

including the existence of the limit (albeit possibly with both sides equal to $+\infty$). Prove that the same holds for general $\mu$ when $\limsup_{p \to \infty} \|f\|_p < \infty$. Assuming that $\mu(X) = 1$, show also that

$$\forall f \in \bigcup_{p > 0} L^p: \lim_{p \to 0} \|f\|_p = \exp\left\{ \int \log |f| \, d\mu \right\},$$

where the integral on the right exists, albeit possibly diverging to $-\infty$.

Problem 3: (Triviality of linear functionals on $L^p$ with $p < 1$) Consider the spaces $\ell^p(N)$ (which are $L^p(N, 2^N, \#)$ for $\#$ denoting the counting measure). Noting that for $p \in (0, 1)$ we have $\ell^p(N) \subseteq \ell^1(N)$, prove that every continuous linear functional on $\ell^p(N)$ extends continuously to $\ell^1(N)$.

Problem 4: Prove the “Principle of condensation of singularities:” Assume that either $p \in [1, \infty)$ or $p = \infty$ with the underlying measure semifinite and let $\{\phi_{ij}: i, j \geq 1\} \subseteq (L^p)^*$ be such that

$$\forall i \geq 1 \exists f_i \in L^p: \sup_{j \geq 1} |\phi_{ij}(f_i)| = \infty$$

Then

$$\exists f \in L^p \forall i \geq 1: \sup_{j \geq 1} |\phi_{ij}(f)| = \infty$$

Problem 5: (Separation of separated convex sets in $L^p$) Let $1 < p < \infty$ and let $C, C' \subseteq L^p$ be two non-empty convex sets such that

$$\inf_{f \in C} \inf_{f' \in C'} \|f - f'\|_p > 0$$

Prove that

$$\exists \phi \in (L^p)^*: \sup_{f' \in C'} \phi(f) < \inf_{f \in C} \phi(f)$$
Problem 6: Let $1 < p < \infty$ and suppose that $\{f_n\}_{n \geq 1} \subseteq L^p$ is such that $\sup_{n \geq 1} \|f_n\|_p < \infty$ and $f_n \to f$ in measure. Prove that $f_n \overset{w}{\to} f$ weakly in $L^p$. (In $\ell^p(\mathbb{N})$, the stated conditions are also necessary for weak convergence.)

Problem 7: Let $\mathcal{V}$ be a normed linear space and let $\mathcal{V}^*$ be the space of continuous linear functionals on $\mathcal{V}$. Do the following:

1. Let $\mathcal{O}$ denote the Euclidean topology of the reals and set

$$\mathcal{T}_0 := \bigcup_{n \geq 1} \left\{ \bigcap_{i=1}^n \phi_i^{-1}(O_i) : \phi_1, \ldots, \phi_n \in \mathcal{V}^*, O_1, \ldots, O_n \in \mathcal{O} \right\}$$

Denote by $\mathcal{T}$ the set of arbitrary unions of sets from $\mathcal{T}_0$. Prove that $\mathcal{T}$ is the weak topology on $\mathcal{V}$.

2. Use this to prove that every weakly open set is (norm)-open.

3. Prove that, if $\mathcal{V}^*$ is infinitely dimensional (meaning: $\mathcal{V}^*$ is not the norm-closure of a finitely-dimensional linear subset thereof), then every weakly open set is unbounded in norm. What can you say if $\mathcal{V}^*$ is finitely dimensional?

Problem 8: Consider the spaces $\ell^p(\mathbb{N})$. Do as follows:

1. For $p \in (1, \infty)$, construct a sequence in $\ell^p(\mathbb{N})$ that converges weakly but that contains no subsequence that converges in norm.

2. Prove that if a sequence in $\ell^1(\mathbb{N})$ converges weakly then it converges in norm. (Normed spaces for which this holds are said to have Schur’s property.)

3. Can you construct a net $\{f_\alpha\}_{\alpha \in I} \subseteq \ell^1(\mathbb{N})$, for $I$ a suitable (necessarily uncountable) directed set, such that $f_\alpha \overset{w}{\to} 0$ yet $\|f_\alpha\|_1 = 1$ for all $\alpha \in I$?