3. ABSOLUTELY CONTINUOUS FUNCTIONS

Having resolved Question 1.4 from Section 1, we will now return to the less abstract setting in Question 1.2. We start by making some useful observations about functions \( F : [a, b] \rightarrow \mathbb{R} \) that satisfy (1.2) for some integrable \( f \). We need:

**Definition 3.1** A function \( F : [a, b] \rightarrow \mathbb{R} \) is said to be **absolutely continuous** (or AC, for short) if for each \( \varepsilon > 0 \) there is \( \delta > 0 \) such that for all \( n \geq 1 \) and all \( a_1, b_1, \ldots, a_n, b_n \in \mathbb{R} \) satisfying

\[
a \leq a_0 < b_1 \leq a_2 < b_1 \leq \cdots \leq a_n < b_n < b
\]

we have

\[
\sum_{i=1}^{n} (b_i - a_i) < \delta \quad \Rightarrow \quad \sum_{i=1}^{n} |F(b_i) - F(a_i)| < \varepsilon.
\]

If we wish to specify the interval on which absolute continuity holds, we say that \( F \) is AC on \([a, b]\).

As we will see in the next proof, this notion is very closely related to the absolute continuity for measures. Indeed, we have:

**Lemma 3.2** If \( F : [a, b] \in \mathbb{R} \) satisfies (1.2) with some integrable \( f \), then \( F \) is AC on \([a, b]\).

**Proof.** Consider the measure

\[
\mu(A) := \int_{A \cap [a, b]} f \, d\lambda.
\]

Then \( \mu \) is finite thanks to integrability of \( f \) and \( \mu \ll \lambda \). By Lemma 1.7, \( \mu \) is AC w.r.t. \( \lambda \). This means

\[
\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall A \in \mathcal{B}(\mathbb{R}) : \ \lambda(A) < \delta \ \Rightarrow \ \mu(A) < \varepsilon.
\]

Applying this \( A := \bigcup_{i=1}^{n} (a_i, b_i) \) for \( a_1, b_1, \ldots, a_n, b_n \) as in (3.2) and noting that

\[
\lambda((a_i, b_i]) = b_i - a_i \quad \text{and} \quad \mu((a_i, b_i]) = F(b_i) - F(a_i)
\]

the claim follows. \( \square \)

Absolute continuity is thus a necessary condition for (1.2) to hold. As it turns out, it is also sufficient:

**Theorem 3.3** (Vitali-Lebesgue) Let \( a < b \) and let \( F : [a, b] \rightarrow \mathbb{R} \). The following are then equivalent:

1. \( F \) is AC on \([a, b]\).
2. \( \exists f \in L^1([a, b], \lambda) \) such that

\[
\forall x \in [a, b] : \quad F(x) - F(a) = \int_{(a, x]} f \, d\lambda.
\]

In light of Lemma 3.2, it remains to prove (1) \( \Rightarrow \) (2). As we will see, the following standard concept will turn out to be very useful:
Definition 3.4 (Total/First variation) Given a function \( F : [a, b] \rightarrow \mathbb{R} \), the quantity
\[
V(F, [a, b]) := \sup_{n \geq 1} \sup_{a = t_0 < t_1 < \cdots < t_n = b} \sum_{i=1}^{n} |F(t_i) - F(t_{i-1})|
\] (3.7)
is the total variation a.k.a. first variation of \( F \) on \([a, b]\). We say that \( F \) is of finite variation on \([a, b]\) if \( V(F, [a, b]) < \infty \).

It is easy to check that, for \( F \) of finite total variation on \([a, b]\), if \( F \) is continuous, then so is \( x \mapsto V(F, [a, x]) \). As it turns out, the AC property is inherited as well:

Lemma 3.5 If \( F : [a, b] \rightarrow \mathbb{R} \) is AC on \([a, b]\) then:

1. \( F \) is uniformly continuous on \([a, b]\),
2. \( F \) is of finite variation, \( V(F, [a, b]) < \infty \).
3. \( x \mapsto V(F, [a, x]) \) is AC on \([a, b]\).

Proof. For (1) we just apply (3.2) with \( n = 1 \). For (2) let \( \delta > 0 \) be such that (3.2) holds with \( \epsilon := 1 \). Then for any interval \([a', b'] \subseteq [a, b]\) with \( b' - a' < \delta \), any partition \( \{a' = t_0 < \cdots < t_n = b'\} \) of \([a', b']\) obeys
\[
\sum_{i=1}^{n} |F(t_i) - F(t_{i-1})| < 1.
\] (3.8)

It follows that, for all intervals \( I \subseteq [a, b] \) with \( |I| < \delta \) we have \( V(F, I) \leq 1 \). Since forcing any given point to occur in the supremum in (3.20) does not change the result, the total variation is additive,
\[
\forall c \in (a, b) : \quad V(F, [a, b]) = V(F, [a, c]) + V(F, [c, b]).
\] (3.9)

By writing \([a, b]\) as a disjoint union of \( n := \lceil \frac{b-a}{\delta} \rceil \) of intervals \( I_1, \ldots, I_n \) of length less than \( \delta \), we get
\[
V(F, [a, b]) = \sum_{i=1}^{n} V(F, I_i) \leq \left\lceil \frac{b-a}{\delta} \right\rceil.
\] (3.10)

In particular, \( F \) is of finite total variation.

For (3), given \( \epsilon > 0 \) let \( \delta > 0 \) be such that (3.2) holds. For any collection of disjoint intervals \( \{a_i, b_i\} \subseteq [a, b] \) subject to \( \sum_{i=1}^{n} (b_i - a_i) < \delta \) and each \( i = 1, \ldots, n \), consider a partition of \([a_i, b_i]\) by points \( a_i = t_{i,0} < \cdots < t_{i,m_i} = b_i \) such that
\[
\sum_{j=1}^{m_i} |F(t_{i,j}) - F(t_{i,j-1})| > V(F, [a_i, b_i]) - \epsilon / n.
\] (3.11)

Then
\[
\sum_{i=1}^{n} \sum_{j=1}^{m_i} (t_{i,j} - t_{i,j-1}) = \sum_{i=1}^{n} (b_i - a_i) < \delta
\] (3.12)
and so the AC property of \( F \) gives
\[
\sum_{i=1}^{n} \sum_{j=1}^{m_i} |F(t_{i,j}) - F(t_{i,j-1})| < \epsilon.
\] (3.13)
It follows that
\[ \sum_{i=1}^{n} V(F, [a_i, b_i]) < 2\epsilon. \]  
(3.14)

Since \( \epsilon > 0 \) is arbitrary, this proves AC property for \( x \mapsto V(F, [a, x]) \) is AC on \([a, b]\). \( \square \)

It is easy to check that if \( F \) is non-decreasing, then it is of finite variation. Since, by the triangle inequality,
\[ V(F_1 + F_2, [a, b]) \leq V(F_1, [a, b]) + V(F_2, [a, b]) \]  
(3.15)
a difference of two non-decreasing functions is of finite variation as well. As it turns out, this is in fact all one can get:

**Proposition 3.6** (Jordan decomposition) Let \( F: [a, b] \to \mathbb{R} \) obey \( V(F, [a, b]) < \infty \). Then there exists \( F_1, F_2: [a, b] \to \mathbb{R} \) non-decreasing such that \( F = F_1 - F_2 \) on \([a, b]\). This decomposition is not unique but we have:

1. If \( F \) is continuous, then both \( F_1 \) and \( F_2 \) can be chosen continuous.
2. If \( F \) is AC, then both \( F_1 \) and \( F_2 \) can be chosen AC.

**Proof.** Define
\[ F_1(x) := V(F, [a, x]) \]  
(3.16)
and
\[ F_2(x) := V(F, [a, x]) - F(x). \]  
(3.17)
Then \( F_1 \) is non-decreasing in light of (3.9) and the non-negativity of the total variation. For the monotonicity of \( F_2 \) we in turn use that, for each \( x \in [a, b] \) and each \( h > 0 \) such that \( x + h \in [a, b] \),
\[ V(F, [a, x + h]) = V(F, [a, x]) + V(F, [x, x + h]) \]
\[ \geq V(F, [a, x]) + |F(x + h) - F(x)| \]  
(3.18)
by restricting the supremum defining \( V(F, [x, x + h]) \).

The decomposition is not unique because adding any non-decreasing function to both \( F_1 \) and \( F_2 \) does not change their difference. But if \( F \) is continuous, then \( F_1 \) and \( F_2 \) in (3.16–3.17) are continuous by Lemma 3.5(2), while if \( F \) is even AC, then \( F_1 \) and \( F_2 \) are AC by Lemma 3.5(3) and the fact that the sum (or difference) of two AC functions is AC. \( \square \)

We remark in passing that, although \( F_1 \) and \( F_2 \) in the above Jordan decomposition of \( F \) are not unique, there are choices — different from (3.16–3.17)! — that are in a well-defined sense minimal possible. Indeed, define the positive and negative variations by
\[ P(F, [a, b]) := \sup_{n \geq 1} \sup_{a = t_0 < t_1 < \cdots < t_n = b} \sum_{i=1}^{n} (F(t_i) - F(t_{i-1}))^+ \]  
(3.19)
and
\[ N(F, [a, b]) := \sup_{n \geq 1} \sup_{a = t_0 < t_1 < \cdots < t_n = b} \sum_{i=1}^{n} (F(t_i) - F(t_{i-1}))^- \]  
(3.20)
where \( a^+ := \max\{0, a\} \) and \( a^- := \max\{0, -a\} \). Then we have:
Lemma 3.7 Let $F: [a, b] \to \mathbb{R}$. Then the quantities $V(F, [a, b]), P(F, [a, b])$ and $N(F, [a, b])$ are either all infinite or all finite. In the latter case, we have
\begin{equation}
V(F, [a, b]) = P(F, [a, b]) + N(F, [a, b])
\end{equation}
and
\begin{equation}
F(b) - F(a) = P(F, [a, b]) - N(F, [a, b]).
\end{equation}
Moreover, $x \mapsto P(F, [a, x])$ and $x \mapsto N(F, [a, x])$ are both non-decreasing and so (3.22) gives a Jordan decomposition of $F$. In addition, if $F = F_1 - F_2$ for non-decreasing $F_1$ and $F_2$ then
\begin{equation}
x \mapsto F_1(x) - P(F, [a, x]) \quad \text{and} \quad x \mapsto F_2(x) - N(F, [a, x])
\end{equation}
are non-decreasing on $[a, b]$.

We leave the proof of this lemma to the reader. Instead, we give:

Proof of Theorem 3.3. Let $F: [a, b] \to \mathbb{R}$ be AC. By Lemma 3.5, $F$ is of finite total variation which is AC as well. By Proposition 3.6, there exist non-decreasing, AC functions $F_1$ and $F_2$ such that $F = F_1 - F_2$. By linearity of the integral, it thus suffices to prove the claim for $F: [a, b] \to \mathbb{R}$ non-decreasing and AC.

Extend $F$ to equal $F(a)$ on $(-\infty, a)$ and to $F(b)$ on $(b, \infty)$. Then $F$ is non-decreasing and continuous on all of $\mathbb{R}$ and so, by Theorem 1.3, there exists a measure $\mu_F$ on $B(\mathbb{R})$ such that
\begin{equation}
\forall x, y \in \mathbb{R}: \quad x < y \Rightarrow F(y) - F(x) = \mu_F((x, y)).
\end{equation}
We claim that $\mu_F \ll \lambda$. Indeed, let $A \in B(\mathbb{R})$ obey $\lambda(A) = 0$. Then, by the outer regularity of the Lebesgue measure, for each $\delta > 0$ there is an open set $O_\delta \subset \mathbb{R}$ with $A \subseteq O_\delta$ and $\lambda(O_\delta) < \delta$. Every open subset of $\mathbb{R}$ is a finite or countable union of disjoint open intervals,
\begin{equation}
O_\delta = \bigcup_{i=1}^{N} (a_i, b_i)
\end{equation}
for some $N \in \mathbb{N} \cup \{\infty\}$ for which we get, by $\sigma$-additivity of $\lambda$,
\begin{equation}
\sum_{i=1}^{n} (b_i - a_i) \leq \sum_{i=1}^{\infty} (b_i - a_i) < \delta
\end{equation}
for all $n \in \mathbb{N}$ with $n \leq N$. Assuming $\epsilon > 0$ is related to $\delta > 0$ as in the definition of the AC property of $F$, hereby we get $\sum_{i=1}^{n} |F(b_i) - F(a_i)| < \epsilon$ for each $n \geq 1$ and so, by taking $n \to \infty$,
\begin{equation}
\mu_F(A) \leq \mu_F(O_\delta) = \sum_{i=1}^{\infty} |F(b_i) - F(a_i)| \leq \epsilon.
\end{equation}
Since this holds for all $\epsilon > 0$, we must have $\mu_F(A) = 0$ thus proving $\mu_F \ll \lambda$.

The Lebesgue-Radon-Nikodym Theorem now gives existence of a measurable function $f: \mathbb{R} \to [0, \infty)$ such that
\begin{equation}
\forall A \in B(\mathbb{R}): \quad \mu_F(A) = \int_A f \, d\lambda
\end{equation}
Using this for $A := (a, x]$ in combination with (3.24), we then get (3.6). Since $\mu_F$ is finite, we have $f \in L^1([a,b], \lambda).$
As the proof shows, the AC property of $F$ is equivalent to to $\mu_F$ being AC w.r.t. $\lambda$. This brings us to the observation that the Lebesgue measure has not played any particular role in the above arguments; indeed, all we needed was an outer-regular measure that was finite on bounded intervals. This means that we can replace the Lebesgue measure by any Radon measure $\nu$ on $\mathbb{R}$. The definition of absolute continuity of a function $F$ will then have to be modified to a relative notion of “AC w.r.t. $\nu$.” This amounts to replacing the sum of interval lengths on the left of (3.2) by the sum of $\sum_{i=1}^{n} \nu((a_i, b_i])$.

Some complications do arise, though. For instance, since the relative notion no longer guarantees that $F$ is continuous, we need to assume right-continuity of $F$ in order to get (3.24). (This does propagate through the Jordan decomposition.) For similar reasons, since $\nu$ may charge singletons, we cannot freely change $(a, x]$ to $[a, x]$ in the integral in (3.6). Note that $F$ being AC w.r.t. $\nu$ implies that $\nu$ puts a non-trivial mass on every discontinuity point of $F$ and the jump of $F$ at that point is smaller the smaller the mass of $\nu$ at that point is.