1. **Lebesgue-Radon-Nikodym Theorem**

The first part of the 245B course will be devoted to the subject of **differentiation**. This is a subject that stems directly from the classical results of calculus:

**Theorem 1.1 (FTC I)** Suppose $f: [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$. Then for all $x \in (a, b)$ such that $f$ is continuous at $x$,

$$\frac{d}{dx} \int_a^x f(z)\,dz = f(x). \quad (1.1)$$

**Theorem 1.2 (FTC II)** Suppose $F: [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on $(a, b)$ with $F'$ Riemann integrable on $[a, b]$. Then

$$F(b) - F(a) = \int_a^b F'(x)\,dx. \quad (1.2)$$

The statement of Theorem 1.2 actually demonstrates one of the basic shortcomings of the Riemann integral: To represent $F$ as integral of its derivative, it is not enough to know that $F$ is differentiable; we need also that the derivative is integrable. Although a derivative of a continuous function cannot generally be too bad, this assumption is not superfluous. Indeed, there is a classical example due to Volterra of a function that is differentiable with derivative bounded but not Riemann integrable. This problem does not occur when Lebesgue integral is used instead.

Leaving FTC I aside for a while, we will start by recasting FTC II in more general terms. Indeed, we ask:

**Question 1.3** For what functions $F: [a, b] \to \mathbb{R}$ is there a measurable $f: [a, b] \to \mathbb{R}$ such that for all $x \in [a, b]$,

$$F(x) - F(a) = \int_{[a,x]} f \, d\lambda, \quad (1.3)$$

where $\lambda$ is the Lebesgue measure.

Note that, if such an $f$ exists, we may regard it as a version of the derivative — defined using integral, rather than differential, calculus. In this lecture, we will ask (and answer) Question 1.3 for the special case of functions $F$ that are monotone and right-continuous. Under this assumption, $F$ is directly linked to a measure. Indeed, recall:

**Theorem 1.4** For each non-decreasing, right-continuous $F: \mathbb{R} \to \mathbb{R}$ there exists a unique Radon measure $\mu_F$ on $\mathbb{R}$ such that

$$\forall a < b: \quad \mu_F([a,b]) = F(b) - F(a) \quad (1.4)$$

Conversely, every Radon measure on $\mathbb{R}$ is linked to a non-decreasing, right-continuous $F$ via (1.4) that is determined uniquely up to the choice of the value at one point.

The representation (1.4) permits us to cast Question 1.3 in the form:
**Question 1.5** For what measures $\mu$ and $\nu$ on a measurable space $(\Omega, \mathcal{F})$ is there a measurable function $f : \Omega \rightarrow [0, \infty)$ such that
\[
\forall E \in \mathcal{F} : \quad \mu(E) = \int_E f \, d\nu. \quad (1.5)
\]

We start by noting that (1.5) cannot hold just for any pair of measures. Indeed, the integral vanishes whenever $\nu(E) = 0$ and hence so must $\mu(E)$. Any pair $\mu$ and $\nu$ satisfying (1.5) must therefore adhere to the following concept:

**Definition 1.6** Given two measures $\mu$ and $\nu$ on $(\Omega, \mathcal{F})$ we define
\[
\mu \ll \nu := \forall E \in \mathcal{F} : \quad \nu(E) = 0 \Rightarrow \mu(E) = 0. \quad (1.6)
\]
This concept is often referred to using the term of absolute continuity although that term is actually somewhat more restrictive:

**Definition 1.7** (Absolute continuity) Given two measures $\mu$ and $\nu$ on $(\Omega, \mathcal{F})$, we say that $\mu$ is absolutely continuous with respect to $\nu$, abbreviated as “$\mu$ is AC w.r.t. $\nu$,” if
\[
\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall E \in \mathcal{F} : \quad \nu(E) < \delta \Rightarrow \mu(E) < \epsilon. \quad (1.7)
\]
That using the two notions exchangeably is no much of a loss is the content of:

**Lemma 1.8** Given two measures $\mu$ and $\nu$ on $(\Omega, \mathcal{F})$ with $\mu$ finite,
\[
\mu \ll \nu \iff \mu \text{ is AC w.r.t. } \nu. \quad (1.8)
\]

**Proof.** The implication $\Leftarrow$ is trivial. Indeed, if $\nu(E) = 0$ then $\nu(E) < \delta$ for all $\delta > 0$ and so $\mu(E) < \epsilon$ for all $\epsilon > 0$. We will prove $\Rightarrow$ by proving the contrapositive. Suppose that $\mu$ is NOT AC w.r.t. $\nu$. Then
\[
\exists \epsilon > 0 \quad \forall n \geq 1 \quad \exists E_n \in \mathcal{F} : \quad \nu(E_n) < 2^{-n} \land \mu(E_n) \geq \epsilon. \quad (1.9)
\]
Then the $\sigma$-additivity of $\nu$ implies
\[
\nu\left(\bigcup_{k \geq n} E_k\right) \leq \sum_{k \geq n} 2^{-k} = 2^{-n+1}. \quad (1.10)
\]
Since $\nu(\bigcup_{k \geq n} E_k) \leq 1$, the Downward Monotone Convergence Theorem for measures applies and we get
\[
\nu\left(\bigcap_{n \geq 1} \bigcup_{k \geq n} E_k\right) = \lim_{n \to \infty} \nu\left(\bigcup_{k \geq n} E_k\right) = 0. \quad (1.11)
\]
On the other hand, since $\mu$ is finite, the Downward Monotone Convergence Theorem for measures gives
\[
\mu\left(\bigcap_{n \geq 1} \bigcup_{k \geq n} E_k\right) = \lim_{n \to \infty} \mu\left(\bigcup_{k \geq n} E_k\right) \geq \limsup_{n \to \infty} \mu(E_n) \geq \epsilon. \quad (1.12)
\]
Hence, NOT $\mu$ is AC w.r.t. $\nu$ implies NOT $\mu \ll \nu$ and so $\Rightarrow$ holds in (1.8). \qed

As it turns out, finiteness of $\mu$ is necessary for Lemma 1.8 to hold in general. Indeed, here is a counterexample: Set $\Omega := \mathbb{N}$ with $\mathcal{F} = 2^\mathbb{N}$ and define $\mu$ and $\nu$ by
\[
\forall n \in \mathbb{N} : \quad \mu(\{n\}) := 1 \land \nu(\{n\}) := 2^{-n}. \quad (1.13)
\]
Then $\mu \ll \nu$ because the only $\nu$-null set is the empty set. Yet $\mu$ is not AC w.r.t. $\nu$.

The condition $\mu \ll \nu$ may be interpreted as the statement that all of the mass of $\mu$ lies on a set that supports all of the mass of $\nu$. If both $\mu \ll \nu$ and $\nu \ll \mu$ hold, then we say that $\mu$ and $\nu$ are mutually absolutely continuous. We will also need notion for the situation that the two measures put their mass on disjoint sets:

**Definition 1.9** (Mutually singular measures)  Given two measures $\mu$ and $\nu$ on $(\Omega, \mathcal{F})$, we define the relation

$$\mu \perp \nu := \exists A \in \mathcal{F}: \mu(A) = 0 = \nu(\Omega \setminus A).$$

(1.14)

We refer to $\mu \perp \nu$ as the property that $\mu$ and $\nu$ are singular with respect to each other or mutually singular.

We have seen that $\mu \ll \nu$ is necessary for (1.5) to hold. What is perhaps surprising is that this condition is also more or less sufficient. This, and a bit more, is the content of the main theorem proved in this lecture:

**Theorem 1.10** (Lebesgue decomposition, Radon-Nikodym Theorem)  Let $\mu, \nu$ be $\sigma$-finite measures on $(\Omega, \mathcal{F})$. Then there are unique measures $\mu_1, \mu_2$ so that

$$\mu = \mu_1 + \mu_2 \land \mu_1 \ll \nu \land \mu_2 \perp \nu.$$  

(1.15)

Moreover, there exists an $\mathcal{F}$-measurable $f : \Omega \to [0, \infty)$ such that

$$\mu_1(E) = \int_E f \, d\nu, \quad E \in \mathcal{F}.$$  

(1.16)

The function $f$ is determined uniquely up to equivalence $\nu$-a.e.

Here we recall that a measure $\mu$ on $(\Omega, \mathcal{F})$ is said to be $\sigma$-finite if there exists a sequence $\{\Omega_n\}_{n \geq 1} \subseteq \mathcal{F}$ of disjoint sets such that

$$\bigcup_{n \geq 1} \Omega_n = \Omega \land \forall n \geq 1: \mu(\Omega_n) < \infty.$$  

(1.17)

The uniqueness of function $f$ in (1.16) leads to:

**Definition 1.11** (Radon-Nikodym derivative)  For $\sigma$-finite measures $\mu$ and $\nu$ satisfying $\mu \ll \nu$, we call the unique $f$ for which (1.5) holds the Radon-Nikodym derivative of $\mu$ with respect to $\nu$, with notation $\frac{d\mu}{d\nu}$.

There is a classical proof of the *Lebesgue decomposition* (1.15) as well as the representation (1.16) that is based on the concept of a signed measure and the Hahn and Jordan decompositions. While we will discuss this proof later, we start with a proof (due to John von Neumann) that is based on functional analysis. The key input is the classical lemma proved in 245A:

**Lemma 1.12** (Riesz representation)  Let $L^2(\Omega, \mathcal{F}, \mu)$ denote the space of square integrable functions on a measure space $(\Omega, \mathcal{F}, \mu)$ and let $\phi : L^2(\Omega, \mathcal{F}, \mu) \to \mathbb{R}$ be a continuous linear functional. Then there exists $h \in L^2(\Omega, \mathcal{F}, \mu)$ such that

$$\forall f \in L^2(\Omega, \mathcal{F}, \mu): \quad \phi(f) = \int h(x)f(x) \, d\mu(x).$$  

(1.18)
With this fact in hand, we can now give:

Proof of Theorem 1.10. The proof comes in three steps:

Step 1 (existence for finite measures): Let us first assume that both \( \mu \) and \( \nu \) are finite. Consider the space \( L^2(\Omega, \mathcal{F}, \mu + \nu) \) and define the linear functional

\[
\phi(f) := \int f(x) \mu(dx), \quad f \in L^2(\Omega, \mathcal{F}, \mu + \nu). \tag{1.19}
\]

Since \( \mu \) is finite, the Cauchy-Schwarz inequality implies

\[
|\phi(f)| \leq \mu(\Omega)^{1/2} \left[ \int f^2 d\mu \right]^{1/2} \leq \mu(\Omega)^{1/2} \|f\|_{L^2(\Omega, \mathcal{F}, \mu + \nu)}. \tag{1.20}
\]

This shows that \( \phi \) is everywhere defined and bounded, and thus continuous. The Riesz representation in Lemma 1.12 ensures the existence of an \( h \in L^2(\Omega, \mathcal{F}, \mu + \nu) \) such that

\[
\int f(x) \mu(dx) = \int f(x) h(x) [\mu + \nu](dx), \quad f \in L^2(\Omega, \mathcal{F}, \mu + \nu). \tag{1.21}
\]

We now make a couple of observations:

Claim 1: \( \nu(h \geq 1) = 0. \)

Indeed, plugging \( f := 1_{\{h \geq 1\}} \) implies \( \mu(h \geq 1) \geq \mu(h \geq 1) + \nu(h \geq 1) \) which yields the claim. Similarly, we observe:

Claim 2: \( \mu(h > 1) = 0. \)

Indeed, taking \( f := 1_{\{h \geq 1+\varepsilon\}} \) yields \( \mu(h \geq 1+\varepsilon) \geq (1+\varepsilon)\mu(h \geq 1+\varepsilon) \) and so we have \( \mu(h \geq 1+\varepsilon) = 0 \) for all \( \varepsilon > 0 \). The claim is obtained by taking a union along a sequence of \( \varepsilon \downarrow 0 \). Finally:

Claim 3: \( [\mu + \nu](h < 0) = 0. \)

Here we take \( f := 1_{\{h < -\varepsilon\}} \) to get \( -\varepsilon[\mu + \nu](h < -\varepsilon) \geq 0 \) forcing \( [\mu + \nu](h < -\varepsilon) = 0 \). Taking a union along a sequence \( \varepsilon \downarrow 0 \) then finishes the proof.

With these observations in hand, we now define

\[
\mu_1(dx) := 1_{\{0 \leq h(x) < 1\}} \mu(dx) \quad \text{and} \quad \mu_2(dx) := 1_{\{h(x) = 1\}} \mu(dx). \tag{1.22}
\]

Then \( \mu_2 \perp \nu \) and \( \mu_1 + \mu_2 = \mu \) by the above claims. For the proof of the remaining properties, we rewrite (1.21) as

\[
\int f(x) [1 - h(x)] \mu(dx) = \int f(x) h(x) \nu(dx), \quad f \in L^2(\Omega, \mathcal{F}, \mu + \nu). \tag{1.23}
\]

On the left hand side we may freely replace \( \mu \) by \( \mu_1 \). Setting

\[
f(x) := 1_{\{0 \leq h(x) < 1 - \varepsilon\}} \frac{1}{1 - h(x)} A \tag{1.24}
\]

which is bounded and thus in \( L^2(\Omega, \mathcal{F}, \mu + \nu) \) for \( \varepsilon > 0 \), and taking \( \varepsilon \downarrow 0 \) with the help of the Monotone Convergence Theorem then yields

\[
\mu_1(A) = \int_A \frac{h}{1 - h} 1_{\{0 \leq h < 1\}} \nu, \quad A \in \mathcal{F}. \tag{1.25}
\]

In particular, we get \( \mu_1(A) = 0 \) if \( \nu(A) = 0 \), so \( \mu_1 \ll \nu \) as claimed.
This proves the existence part of the statement for \( \mu \) and \( \nu \) finite. It remains to prove the uniqueness of the decomposition in (1.15) and \( \nu \)-a.e. uniqueness of the Radon-Nikodym derivative \( f \) in (1.16) and then extend these to \( \sigma \)-finite measures.

**Step 2 (uniqueness):** Keeping the assumption that \( \mu \) and \( \nu \) are finite, suppose \((\mu_1, \mu_2)\) and \((\tilde{\mu}_1, \tilde{\mu}_2)\) are two pairs of measures for which (1.15) holds. Then there is \( A \in \mathcal{F} \) be such that \( \mu_2(\Omega \setminus A) = 0 = \nu(A) \) and \( A' \in \mathcal{F} \) such that \( \tilde{\mu}_2(\Omega \setminus A') = 0 = \nu(A') \). But then also \( A \cup A' \) is \( \nu \)-null and \( \Omega \setminus (A \cup A') \) is null for both \( \mu_2 \) and \( \tilde{\mu}_2 \), and so we may suppose \( A' = A \) without loss of generality. Then \( \mu_1 \ll \nu \) and \( \tilde{\mu}_1 \ll \nu \) imply

\[
\mu_2(E) = \mu(E \cap A) = \tilde{\mu}_2(E), \quad E \in \mathcal{F},
\]

and, since \( \mu \) is finite, also

\[
\mu_1(E) = \mu(E) - \mu_2(E) = \mu(E) - \tilde{\mu}_2(E) = \tilde{\mu}_1(E), \quad E \in \mathcal{F}.
\]

This proves uniqueness of the Lebesgue decomposition (1.15). For the uniqueness in (1.16), note that if \( f \) and \( \tilde{f} \) are two \( \mathcal{F} \)-measurable functions such that (1.16) holds, then

\[
\int_E (f - \tilde{f}) \, d\nu = 0, \quad E \in \mathcal{F}.
\]

Taking \( E := \{ f > \tilde{f} \} \) shows \( \nu(f > \tilde{f}) = 0 \) and, by symmetry, \( \nu(f \neq \tilde{f}) = 0 \). Hence, \( f \) and \( \tilde{f} \) are equal \( \nu \)-a.e.

**Step 3 (extension to \( \sigma \)-finite measures):** Let us assume that \( \mu \) and \( \nu \) are only \( \sigma \)-finite. Using pairwise intersections to refine the sequences in (1.17) for \( \mu \) and \( \nu \) to a common sequence, there is a sequence \( \{\Omega_n\}_{n \geq 1} \subseteq \mathcal{F} \) of disjoint sets such that \( \bigcup_{n \geq 1} \Omega_n = \Omega \) and

\[
\forall n \geq 1 : \quad \mu(\Omega_n) < \infty \land \nu(\Omega_n) < \infty.
\]

Applying the above to finite measures \( \nu(\Omega_n \cap \cdot) \) and \( \nu(\Omega_n \setminus \cdot) \), for each \( n \geq 1 \) we get existence of measure \( \kappa_n \) with \( \kappa_n(\Omega \setminus \Omega_n) = 0 \) and \( \kappa_n \perp \nu \) and functions \( f_n : \Omega \to [0, \infty) \) with \( f_n = 0 \) on \( \Omega \setminus \Omega_n \) such that

\[
\forall n \geq 1 \ \forall E \in \mathcal{F} : \quad \mu(E \cap \Omega_n) = \kappa_n(E) + \int_E f_n \, d\nu
\]

Define a measure \( \kappa \) and a function \( f : \Omega \to [0, \infty) \) by

\[
\kappa(E) := \sum_{n \geq 1} \kappa_n(E) \quad \text{and} \quad f(x) := \sum_{n \geq 1} f_n(x).
\]

Then \( \kappa \perp \nu \) and, by the Monotone Convergence Theorem,

\[
\forall E \in \mathcal{F} : \quad \mu(E) = \kappa(E) + \int_E f \, d\nu.
\]

Restricting back to subsets of \( \Omega_n \) shows, via the above uniqueness argument, that measure \( \kappa \) is unique and function \( f \) is unique up to modifications on \( \nu \)-null sets.

We close this section by noting that Theorem 1.10 fails in general if \( \nu \) is NOT \( \sigma \)-finite. Indeed, consider a measurable space \((\Omega, \mathcal{F})\) with \( \mathcal{F} \) containing all singletons and define \( \nu(A) := \infty \) unless \( A = \emptyset \) for which we set \( \nu(\emptyset) = 0 \). Then \( \mu \ll \nu \) for any measure \( \mu \) on \((\Omega, \mathcal{F})\). If (1.5) were to hold, then taking \( E := \{x\} \) would force \( \mu(\{x\}) = \infty \) unless \( f(x) = 0 \). So if \( \mu \) is finite on all singletons, then \( f = 0 \) and so \( \mu \) must vanish identically. This rules out all (nontrivial) finite \( \mu \).
Similarly, we show that Theorem 1.10 fails in general if just $\mu$ is NOT $\sigma$-finite. For this let $\Omega := [0, 1]$ with $\mathcal{F}$ consisting of countable subsets of $\Omega$ and complements thereof. Let $\nu$ vanish on countable sets and equal one on all others. If (1.5) were in force for some $\mathcal{F}$-measurable $f$ then $\{f \leq a\}$ can be either countable, in which case $\mu(f \leq a) = 0$, or co-countable, in which case $\mu(f \leq a) \leq a < \infty$. But $(\Omega, \mathcal{F})$ supports a (NOT $\sigma$-finite) measure $\mu$ that vanishes on countable sets and diverges on complements thereof. Note that then $\mu \ll \nu$. Under (1.5), we must have $\mu(f \leq a) = 0$ for all $a \in \mathbb{R}$, meaning that $\{f \leq a\}$ is countable for all $a \in \mathbb{R}$, in contradiction with assumed finiteness of $f$. 