HW#4: due Mon 3/2/2020

Problem 1: (Irrational rotations of a circle) Let $\Omega := [0, 1)$, $\mathcal{F} := \mathcal{B}([0, 1))$ and let $\lambda$ be a Lebesgue measure. Given $\alpha \in [0, 1)$, define $\varphi : \Omega \to \Omega$ by $\varphi(x) := (x + \alpha) \mod 1$. Prove that $\varphi$ is a measure preserving transformation. Then, assuming that $\alpha \not\in \mathbb{Q}$, prove that for all $f \in L^1(\Omega, \mathcal{F}, \lambda)$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ \varphi^k = \int_{[0,1)} f \, d\lambda, \quad \lambda\text{-a.e.}$$

Note that this fails when $\alpha$ is rational.

Problem 2: (Density points) Let $\lambda$ be the Lebesgue measure on $\mathbb{R}$ and let $\mathcal{L}(\mathbb{R})$ be the $\sigma$-algebra of Lebesgue measurable sets and $\mathcal{B}(\mathbb{R})$ the $\sigma$-algebra of Borel sets. Abbreviate $B(x, r) := (x - r, x + r)$ and call $x \in \mathbb{R}$ is a density point of $E \in \mathcal{L}(\mathbb{R})$ if

$$\lim_{r \to 0} \frac{\lambda(E \cap B(x, r))}{\lambda(B(x, r))} = 1.$$ 

Let $L(E) := \{x \in \mathbb{R} : x \text{ is density point of } E\}$. Prove the following:

1. $\forall A \in \mathcal{L}(\mathbb{R}) : L(A) \in \mathcal{B}(\mathbb{R})$.
2. $\forall A, B \in \mathcal{L}(\mathbb{R}) : \lambda(A \triangle B) = 0 \Rightarrow L(A) = L(B)$.
3. $\forall A, B \in \mathcal{L}(\mathbb{R}) : L(A \cap B) = L(A) \cap L(B)$.

Also, find $A, B$ disjoint such that $L(A \cup B) \neq L(A) \cup L(B)$. Letting $\mathcal{M} := \{L(E) : E \in \mathcal{L}([0, 1])\}$ and $\rho(A, B) := \lambda(A \triangle B)$ prove that $(\mathcal{M}, \rho)$ is a complete and separable metric space. Note: $L$ is called a lifting because it "lifts" a member from each equivalence class of measurable sets.

Problem 3: (Approximate continuity) A function $f : \mathbb{R} \to \mathbb{R}$ is approximately continuous at $x$ if there is $E \in \mathcal{L}(\mathbb{R})$ such that $x$ is a density point of $E$ and $\lim_{z \to x, z \in E} f(z) = f(x)$. Do as follows:

1. Prove that each $f \in L^1(\lambda)$ is approximately continuous $\lambda$-a.e.
2. Prove that each measurable $f : \mathbb{R} \to \mathbb{R}$ is approximately continuous $\lambda$-a.e.
3. Construct an $f$ that is approximately continuous everywhere but not continuous.
4. Prove that if $f$ is approximately continuous everywhere, then $f$ is a pointwise limit of continuous functions (and thus has many continuity points).

Problem 4: (Density topology) Let $\mathcal{O}$ denote the collection of open subsets of $\mathbb{R}$ and let

$$\mathcal{T} := \left\{ E \in \mathcal{L}(\mathbb{R}) : (\forall x \in E : x \text{ is density point of } E) \right\} \quad (1)$$

(or, equivalently, $\mathcal{T} := \{ E \in \mathcal{L}(\mathbb{R}) : E \subseteq L(E) \}$.) Prove the following:

1. $\mathcal{T}$ is a topology on $\mathbb{R}$ — meaning that $\mathcal{T}$ contains $\emptyset, \mathbb{R}$ and is closed under arbitrary unions and finite intersections — and
2. $f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{O})$ is continuous — meaning $\forall O \in \mathcal{O} : f^{-1}(O) \in \mathcal{T}$ — if and only if $f$ is approximately continuous.
Note: $\mathcal{T}$ is called the *density topology*. This topology refines the usual topology, $\mathcal{O} \subseteq \mathcal{T}$.

**Problem 5**: (Extension of Lebesgue measure) The Lebesgue measure $\lambda$ on $\mathbb{R}$ is defined canonically for all $A \subseteq \mathbb{R}$ open and, by complementation, also for $A \subseteq \mathbb{R}$ closed and bounded. The outer/inner Lebesgue measure of $A \subseteq [0, 1]$ are then given by

\[
\lambda^*(A) := \inf \{ \lambda(O) : A \subseteq O, \text{ open} \}
\]
\[
\lambda_*(A) := \sup \{ \lambda(C) : C \subseteq A, \text{ closed} \}.
\]

Prove that $\lambda^*(A) = \lambda_*(A)$ for all $A \in \mathcal{L}([0, 1])$. Suppose now that $E \subseteq [0, 1]$ is not Lebesgue measurable. Let $\mathcal{F} := \sigma(\{E\} \cup \mathcal{L}([0, 1]))$. Show that then

\[
\mathcal{F} = \{(A \cap E) \cup (B \cap E^c) : A, B \in \mathcal{L}([0, 1])\}
\]

Set, for $A, B \in \mathcal{L}([0, 1])$,

\[
\bar{\lambda}((A \cap E) \cup (B \cap E^c)) := \lambda^*(A \cap E) + \lambda_*(B \cap E^c)
\]

Prove that $\bar{\lambda}$ is a measure on $\mathcal{F}$ that extends $\lambda$ on $\mathcal{L}([0, 1])$.

**Problem 6**: Given a non-empty set $X$, let $\mathcal{S} \subseteq 2^X$ be a semialgebra — i.e., a class of sets that is closed under finite intersections, contains $\emptyset$ and $X$ and with $A \in \mathcal{S}$ implying that $A^c$ is a finite disjoint union of elements from $\mathcal{S}$. Do the following:

1. Prove that $A := \left\{ \bigcup_{i=1}^n A_i : \{A_i\}_{i=1}^n \subseteq \mathcal{S} \wedge \text{disjoint} \right\}$ is an algebra on $X$.
2. Given a function $\mu_0 : \mathcal{S} \to [0, \infty]$ that is
   - finitely additive on $\mathcal{S}$:
     \[
     \forall n \geq 1 \forall A_1, \ldots, A_n \in \mathcal{S} : \text{disjoint } \Rightarrow \bigcup_{i=1}^n A_i \in \mathcal{S} \Rightarrow \mu_0 \left( \bigcup_{i=1}^n A_i \right) = \sum_{i=1}^n \mu_0(A_i)
     \]
   - countably subadditive on $\mathcal{S}$:
     \[
     \forall \{A_i\}_{i \geq 1} \subseteq \mathcal{S} : \bigcup_{i \geq 1} A_i \in \mathcal{S} \Rightarrow \mu_0 \left( \bigcup_{i \geq 1} A_i \right) \leq \sum_{i \geq 1} \mu_0(A_i)
     \]
   prove that $\sum_{i=1}^n \mu_0(A_i) = \sum_{j=1}^m \mu_0(B_j)$ whenever the disjoint unions $\bigcup_{i=1}^n A_i$ and $\bigcup_{j=1}^m B_j$ (of sets from $\mathcal{S}$) are equal. For the disjoint union $\bigcup_{i=1}^n A_i$, we then set
   \[
   \bar{\mu}_0 \left( \bigcup_{i=1}^n A_i \right) := \sum_{i=1}^n \mu_0(A_i).
   \]

   Prove that $\bar{\mu}_0$ is finitely additive and countably subadditive on $\mathcal{A}$ with $\bar{\mu}_0(A) = \mu_0(A)$ whenever $A \in \mathcal{S}$.

*Note: The Hahn-Kolmogorov Theorem then extends $\bar{\mu}_0$ further to a measure on $\sigma(\mathcal{S})$. 
