**Problem 1:** Let \( \nu \) be a finite signed measure on \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) with \( \nu(\{a\}) = 0 \) and define \( F(x) := \nu([a,x]) \). Consider the Jordan decomposition \( \nu = \nu^+ - \nu^- \). Prove that the positive/negative variations of \( F \) obey

\[
P(F, [a,x]) = \nu^+ ([a,x]) \quad \text{and} \quad N(F, [a,x]) = \nu^- ([a,x])
\]

for all \( x \in [a,b] \).

**Problem 2:** Let \( F, G : [a,b] \to \mathbb{R} \) be continuous and of finite variation on \( [a,b] \). Define \( \mu_F \) and \( \mu_G \) to be the associated signed measures defined via

\[
\mu_F((x,y]) = F(y) - F(x)
\]

for \( a \leq x < y \leq b \), and similarly for \( \mu_G \). Prove the integration-by-parts formula:

\[
\int_{[a,b]} F \, d\mu_G + \int_{[a,b]} G \, d\mu_F = F(b)G(b) - F(a)G(a)
\]

Check that the same holds if \( G \) is just right continuous.

**Problem 3:** Let \( f \in L^1(\mathbb{R}, \lambda) \). Show that the Hardy-Littlewood maximal function,

\[
f^*(x) := \sup_{h>0} \frac{1}{h} \int_{[x,x+h]} |f| \, d\lambda
\]

obeys \( f^* \not\in L^1(\mathbb{R}, \lambda) \) unless \( f = 0 \) \( \lambda \)-a.e. Nonetheless, show that

\[
\frac{f^*}{(\log f^*)^{1+\epsilon}} 1_{\{f^*>2\}} \in L^1(\mathbb{R}, \lambda)
\]

for all \( \epsilon > 0 \). Prove also that, for all \( p \in (1, \infty] \),

\[
f \in L^p(\mathbb{R}, \lambda) \quad \Rightarrow \quad f^* \in L^p(\mathbb{R}, \lambda)
\]

In fact, we have \( \|f^*\|_{L^p} \leq c\|f\|_{L^p} \) for some constant \( c = c(p) \).

**Problem 4:** Use the Vitali covering lemma to give a proof of the Hardy-Littlewood maximal inequality for the Lebesgue measure.

**Problem 5:** Prove the following version of the Lebesgue differentiation theorem: For each \( f \in L^1(\mathbb{R}, \lambda) \) and \( \lambda \)-a.e. \( x \in \mathbb{R} \),

\[
\lim_{h \downarrow 0} \int_{[x,x+h]} |f - f(x)| \, d\lambda = 0
\]

(You may cite theorems from class for this.) Prove also that the set of \( x \in \mathbb{R} \) where the limit exists and takes values larger than \( a \geq 0 \) is measurable.

**Problem 6:** Let \( M \in [0, \infty) \). Show that \( F : [a,b] \to \mathbb{R} \) satisfies the Lipschitz condition

\[
\forall x, y \in [a,b]: \quad |F(y) - F(x)| \leq M|x - y|
\]

if and only if \( F \) is AC with \( |F'| \leq M \) \( \lambda \)-a.e.
Problem 7: Construct a set $A \subseteq B([0,1])$ such that $0 < \lambda(A \cap I) < \lambda(I)$ for every interval $I \subseteq [0,1]$. Then define $F, G : [0,1] \to [0,1]$ by
$$F(x) := \lambda([0,x] \cap A) \quad \text{and} \quad G(x) := 2\lambda([0,x] \cap A) - x$$

Prove that $F$ and $G$ are AC, in fact, Lipschitz continuous and yet

1. $F$ is strictly increasing while $\lambda(\{F' = 0\}) > 0$, and
2. $G$ is not monotone on any interval $I \subseteq [0,1]$.

Problem 8: Let $F_n : [a,b] \to [0,\infty)$, for each $n \geq 1$, be non-decreasing continuous such that $G(x) := \sum_{n \geq 1} F_n(x) < \infty$ for all $x \in [a,b]$. Prove that $G'(x)$ exists and
$$G'(x) = \sum_{n \geq 1} F'(x)$$
for $\lambda$-a.e. $x \in (a,b)$. (The continuity requirement can actually be omitted.)

Problem 9: Let $F : \mathbb{R} \to \mathbb{R}$ be Borel (or Lebesgue) measurable and let
$$Z := \{x \in \mathbb{R} : F'(x) \text{ exists and } F'(x) = 0\}.$$ 

Prove that $\lambda(F(Z)) = 0$. Use this to show that if $F'(x)$ exists with $F'(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$ and $F' = 0$ $\lambda$-a.e., then $F$ is constant. (Compare with the Cantor function where the derivative exists and vanishes $\lambda$-a.e. and yet the function is not constant.) Hint: Do not just plug into a lemma from class; use a suitable Vitali cover to prove this directly.

Problem 10: Suppose $F : [a,b] \to \mathbb{R}$ is continuous and such that $F'(x)$ exists (finitely) for all $x \in (a,b) \setminus B$ where $B$ is finite or countable. Prove that
$$F \text{ is AC } \iff V(F,[a,b]) < \infty \iff F' \in L^1([a,b],\lambda)$$

Hint: Prove that such an $F$ obeys Lusin’s condition (N). (The Cantor function shows that this fails for $B$ of the cardinality of the continuum.) Hint: Start with $B = \emptyset$.

Problem 11: Construct an example of a continuous $F : [0,1] \to \mathbb{R}$ that is everywhere differentiable on $(0,1)$ and yet not of finite variation on any subinterval of the form $[0,x]$ with $x \in (0,1]$.