**HW#2: due Mon 1/27/2020**

**Problem 1:** Let $\nu$ be a measure on $(X, \mathcal{F})$. A family $\mathcal{M}$ of measures on $(X, \mathcal{F})$ is said to be uniformly absolutely continuous (AC) with respect to $\nu$ if

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall \mu \in \mathcal{M} \ \forall A \in \mathcal{F}: \ \nu(A) < \delta \ \Rightarrow \ \mu(A) < \epsilon.$$ 

Given a finite measure space $(X, \mathcal{F}, \nu)$ and a non-empty set $B \subseteq L^1(X, \mathcal{F}, \nu)$, consider the measures $\mu_f(A) := \int_A |f| \, d\nu$, $f \in B$. Show that

$$B \text{ is uniformly integrable } \iff \{ \mu_f: f \in B \} \text{ is uniformly AC w.r.t. } \nu \wedge \sup_{f \in B} \mu_f(X) < \infty \quad (1)$$

Here $B$ is uniformly integrable on a finite measure space if

$$\forall \epsilon > 0 \ \exists M > 0: \sup_{f \in B} \int_{\{|f| > M\}} |f| \, d\nu < \epsilon$$

(The definition of uniform integrability on infinite measure spaces is more complicated; one also needs to impose $\sup_{f \in B} \int_{\{|f| < 1/M\}} |f| \, d\nu < \epsilon$ and $\sup_{f \in B} \int |f| \, d\nu < \infty$.)

**Problem 2:** ($L^1$-convergence and uniform integrability) Let $\{f_n\}_{n \geq 1} \subseteq L^1(\mu)$ for $\mu$ finite. Prove that if $f_n \to f$ pointwise $\mu$-a.e., then

$$f_n \to f \text{ in } L^1(\mu) \iff \{f_n\}_{n \geq 1} \text{ is uniformly integrable}$$

(The same is TRUE with the corresponding more involved definition of uniform integrability on infinite measure spaces.)

**Problem 3:** (Abstract change-of-variables formula) If $\mu$ and $\nu$ are $\sigma$-finite measures with $\mu \ll \nu$, the Radon-Nikodym theorem ensures that $\mu = f \nu$. We call this $f$ the Radon-Nikodym derivative of $\mu$ with respect to $\nu$ with notation $\frac{d\mu}{d\nu}$. Prove that

$$\forall g \in L^1(\mu): \quad g \frac{d\mu}{d\nu} \in L^1(\nu) \wedge \int g \, d\mu = \int g \frac{d\mu}{d\nu} \, d\nu$$

**Problem 4:** (Conditional expectation) Let $\nu$ be a finite measure on a measurable space $(\Omega, \mathcal{F})$ and let $f \in L^1(\nu)$. Given a $\sigma$-algebra $\mathcal{G} \subseteq \mathcal{F}$, prove the existence of a $\mathcal{G}$-measurable function $f_\mathcal{G}$ such that

$$\forall E \in \mathcal{G}: \quad \int_E f_\mathcal{G} \, d\nu = \int_E f \, d\nu$$

Prove that this $f_\mathcal{G}$ is unique up to changes on $\nu$-null sets. Show that $f_\mathcal{G} \in L^1(\nu)$ and that, for every $f \in L^1(\nu)$,

$$\{ f_\mathcal{G}: \mathcal{G} \subseteq \mathcal{F}, \sigma\text{-algebra} \} \text{ is uniformly integrable}$$

In probability (where $\nu$ is of total mass one) we call $f_\mathcal{G}$ the conditional expectation of $f$ given $\mathcal{G}$, with notation $E(f|\mathcal{G})$. 
**Problem 5:** (Conditional expectation does not preserve pointwise convergence) Using the notation from the previous problem (with $\nu$ finite), do as follows:

1. Prove that if $f_n \to f$ in $L^1(\nu)$ then also $(f_n)_G \to f_G$ in $L^1(\nu)$, uniformly in the $\sigma$-algebra $G \subseteq \mathcal{F}$. Hint: Show that $f \mapsto f_G$ is a contraction, meaning $\|f_G\|_{L^1(\nu)} \leq \|f\|_{L^1(\nu)}$.
2. Give an example of $\{f_n\}_{n \geq 1} \subseteq L^1(\nu)$ and $G$ such that $f_n$ converges pointwise and in $L^1(\nu)$ to some $f \in L^1(\nu)$ yet $(f_n)_G$ does NOT converge to $f_G$ pointwise a.e.

**Problem 6:** (Radon-Nikodym derivative is a derivative) Let $\mu, \nu$ be finite measures on some measurable space $(X, \mathcal{F})$. Construct a family $\{P_t : t \geq 0\} \subset \mathcal{F}$ of sets such that (defining $N_t := X \setminus P_t$),

(a) $(P_t, N_t)$ is a Hahn decomposition of $X$ with respect to the signed measure $\mu - t\nu$,

(b) $t \mapsto P_t$ is non-increasing, i.e., $0 \leq s < t$ implies $P_t \subset P_s$.

Prove that if $\mu \ll \nu$, then for $A_{k,n} := P_{k2^{-n}} \setminus P_{(k+1)2^{-n}}$,

$$\sum_{k \geq 0} k2^{-n}1_{A_{k,n}} \to \frac{d\mu}{d\nu} \quad \nu\text{-a.e.}$$

*Hint:* Observe that (assuming $\mu \ll \nu$) $\{\frac{d\mu}{d\nu} \geq t\} = P_t \ \nu\text{-a.e.}

**Problem 7:** (Operations with Radon-Nikodym derivative) Let $\mu, \nu, \kappa$ be $\sigma$-finite measures on some measurable space. Prove that:

1. if $\mu \ll \nu$ and $\nu \ll \kappa$ then $\mu \ll \kappa$ and

$$\frac{d\mu}{d\kappa} = \frac{d\mu}{d\nu} \frac{d\nu}{d\kappa} \quad \kappa\text{-a.e.}$$

2. if $\mu$ and $\nu$ are both finite and $\mu \ll \nu$ and $\nu \ll \mu$, then

$$0 < \frac{d\mu}{d\nu} < \infty \quad \text{and} \quad \frac{d\nu}{d\mu} = \left(\frac{d\mu}{d\nu}\right)^{-1} \quad \nu\text{-a.e.}$$

3. if $\mu \ll \nu$ and $\kappa := \mu + \nu$ then $\mu \ll \kappa$ and $f := \frac{d\mu}{d\kappa}$ obeys

$$0 \leq f < 1 \quad \nu\text{-a.e.} \quad \text{and} \quad \frac{d\mu}{d\nu} = \frac{f}{1-f} \quad \nu\text{-a.e.}$$

**Problem 8:** (Radon-Nikodym for product measures) Show that if $\mu_1, \nu_1$ are $\sigma$-finite measures on $(X, \mathcal{F})$ and $\mu_2, \nu_2$ are $\sigma$-finite measures on $(Y, \mathcal{G})$, then $\mu_1 \ll \nu_1$ and $\mu_2 \ll \nu_2$ implies

$$\mu_1 \otimes \mu_2 \ll \nu_1 \otimes \nu_2$$

and

$$\frac{d(\mu_1 \otimes \mu_2)}{d(\nu_1 \otimes \nu_2)}(x, y) = \frac{d\mu_1}{d\nu_1}(x) \frac{d\mu_2}{d\nu_2}(y) \quad \text{for} \ \nu_1 \otimes \nu_2\text{-a.e. } (x, y)$$

Here $\mu_1 \otimes \mu_2$ is the product of $\mu_1$ and $\mu_2$ on the product $\sigma$-algebra $\mathcal{F} \otimes \mathcal{G}$.