46. APPLICATIONS OF UNIFORM CONVERGENCE

We will now move to some more advanced applications of uniform convergence to integration and differentiation theory. We will then apply these to power series and use them to finally define a number of important transcendental functions.

46.1 Integration and differentiation.

Our first application is to convergence of Riemann integrals. The so called Osgood Bounded Convergence Theorem states that if $\{f_n\}_{n\in\mathbb{N}}$ are uniformly bounded functions $[a,b]\to\mathbb{R}$ and $f\colon [a,b]\to\mathbb{R}$ is such that $f_n\to f$ pointwise *and* f is Riemann integrable, then $\int_a^b f_n(x)\mathrm{d}x \to \int_a^b f(x)\mathrm{d}x$. This theorem is rather difficult to prove (which is why we omitted its discussion earlier), but it becomes much easier to make once we replace pointwise convergence by uniform convergence:

Theorem 46.1 *Let $a<b$ be reals and $\{f_n\}_{n\in\mathbb{N}}$ and f functions $[a,b]\to\mathbb{R}$ such that*

$$\Bigl(\forall n\in\mathbb{N}\colon f_n \text{ Riemann integrable}\Bigr)\wedge f_n\to f \text{ uniformly} \tag{46.1}$$

Then

$$f \text{ Riemann integrable } \wedge \lim_{n\to\infty}\int_a^b f_n(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x \tag{46.2}$$

Proof. Notice that the additivity of the integral and Lemma 36.9 give

$$\left|\int_a^b f_m(x)\mathrm{d}x-\int_a^b f_n(x)\mathrm{d}x\right| = \left|\int_a^b (f_n-f_m)(x)\mathrm{d}x\right| \leqslant (b-a)\sup_{x\in[a,b]}\bigl|f_n(x)-f_m(x)\bigr| \tag{46.3}$$

By Lemma 45.7, $f_n\to f$ implies that the right-hand side is smaller than $\epsilon>0$ once m and n are sufficiently large. It follows that

$$\left\{\int_a^b f_n(x)\mathrm{d}x\right\}_{n\in\mathbb{N}} \text{ is Cauchy} \tag{46.4}$$

and since this is a real-valued sequence and $\mathbb{R}$ is complete,

$$L := \lim_{n\to\infty}\int_a^b f_n(x)\mathrm{d}x \ \text{ exists.} \tag{46.5}$$

In particular, given $\epsilon>0$, there is $n_0\in\mathbb{N}$ such that

$$\forall n\geqslant n_0\colon \left|\int_a^b f_n(x)\mathrm{d}x - L\right| < \epsilon \tag{46.6}$$

The uniform convergence then shows existence of $n\geqslant n_0$ such that

$$\sup_{x\in[a,b]}\bigl|f_n(x)-f(x)\bigr| < \frac{\epsilon}{b-a} \tag{46.7}$$

The assumed Riemann integrability of f_n implies that there exists $\delta>0$ such that for any marked partition Π of $[a,b]$,

$$\|\Pi\|<\delta \ \Rightarrow\ \left|R(f_n,\Pi)-\int_a^b f_n(x)\mathrm{d}x\right| < \epsilon \tag{46.8}$$

The linearity of $f \mapsto R(f,\Pi)$ in turn gives

$$\bigl|R(f,\Pi)-R(f_n,\Pi)\bigr| \leqslant (b-a)\sup_{x\in[a,b]}\bigl|f_n(x)-f(x)\bigr| < \epsilon \tag{46.9}$$

where the second inequality follows from (46.7). The triangle inequality then shows that, for each marked partition Π of $[a,b]$ with $\|\Pi\|<\delta$, we have

$$\begin{aligned}\bigl|R(f,\Pi)-L\bigr| &\leqslant \bigl|R(f,\Pi)-R(f_n,\Pi)\bigr| \\ &\quad+\Bigl|R(f_n,\Pi)-\int_a^b f_n(x)\mathrm{d}x\Bigr| + \Bigl|\int_a^b f_n(x)\mathrm{d}x - L\Bigr| < 3\epsilon\end{aligned} \tag{46.10}$$

thus proving that f is Riemann integrable and the integral of f equals L. □

The next application is to differentiation:

Theorem 46.2 *Given real numbers $a<b$, let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of differentiable functions $(a,b)\to\mathbb{R}$ such that*

$$\exists x_0\in(a,b)\colon\ \lim_{n\to\infty} f_n(x_0) \text{ exists} \tag{46.11}$$

and

$$\{f_n'\}_{n\in\mathbb{N}} \text{ is uniformly Cauchy} \tag{46.12}$$

Then there exists a differentiable function $f\colon(a,b)\to\mathbb{R}$ such that

$$f_n\to f \text{ uniformly } \wedge\ f_n'\to f' \text{ uniformly} \tag{46.13}$$

In particular, the derivative can be exchanged with the limit $n\to\infty$.

Proof. For each $n\in\mathbb{N}$ define $\phi_n\colon(a,b)\to\mathbb{R}$ by

$$\phi_n(x) := \begin{cases}\frac{f_n(x)-f_n(x_0)}{x-x_0}, & \text{if } x\neq x_0,\\ f_n'(x_0), & \text{if } x = x_0.\end{cases} \tag{46.14}$$

Since f_n is continuous (being differentiable) and the derivative at x_0 exists, ϕ_n is continuous on (a,b). Next observe that, for $x\neq x_0$, we have

$$\phi_n(x)-\phi_m(x) = \frac{[f_n(x)-f_m(x)]-[f_n(x_0)-f_m(x_0)]}{x-x_0} \tag{46.15}$$

Langrange's Mean-Value Theorem therefore shows

$$\forall x\in(a,b)\,\exists\xi\in(a,b)\colon\ \phi_n(x)-\phi_m(x) = f_n'(\xi)-f_m'(\xi) \tag{46.16}$$

where for $x=x_0$ the claim follows directly from (46.14). Since $\{f_n'\}_{n\in\mathbb{N}}$ is uniformly Cauchy, so is therefore $\{\phi_n\}_{n\in\mathbb{N}}$. By Lemma 45.7, there exists $\phi\colon(a,b)\to\mathbb{R}$ such that $\phi_n\to\phi$ uniformly. Being a uniform limit of continuous functions, ϕ is continuous.

Define $f\colon(a,b)\to\mathbb{R}$ by

$$f(x) := \lim_{n\to\infty} f_n(x_0) + \phi(x)(x-x_0). \tag{46.17}$$

Since $f_n(x) = f_n(x_0)+\phi_n(x)(x-x_0)$, the uniform convergence $\phi_n\to\phi$ implies uniform convergence $f_n\to f$. In particular, the assumption (46.11) holds for *all* $x_0\in(a,b)$. It therefore suffices to prove differentiability of f at x_0 and the pointwise convergence

$f_n'(x_0) \to f'(x_0)$. (Indeed, Corollary 45.8 then upgrades this to uniform convergence.) For this note that, whenever $x \neq x_0$, we have

$$\begin{aligned} \frac{f(x)-f(x_0)}{x-x_0} - \lim_{n\to\infty} f_n'(x_0) &= \lim_{n\to\infty}\left[\frac{f_n(x)-f_n(x_0)}{x-x_0} - f_n'(x_0)\right] \\ &= \lim_{n\to\infty}\bigl[\phi_n(x)-\phi_n(x_0)\bigr] = \phi(x)-\phi(x_0) \end{aligned} \tag{46.18}$$

where we used the definition of ϕ_n in the first line and the convergence $\phi_n \to \phi$ in the second line. The continuity of ϕ implies that the right-hand side tends to zero as $x \to x_0$. So $f'(x_0)$ exists and equals the limit of $f_n'(x_0)$, as desired. □

We remark that in some texts the above is stated under stronger assumptions on the derivatives such as continuity or Riemann integrability. (For instance, if we assume that f_n' is Riemann integrable on $[a,b]$ for each $n \in \mathbb{N}$, the claim could be deduce using the Fundamental Theorem of Calculus and Theorem 46.1.) Our version is the best possible as the assumptions (46.11–46.12) are also necessary for the conclusions.

46.2 Power series and transcendental functions.

We now move to the positive side of the story by demonstrating its application to functions given as a convergent power series. Theorem 46.1 allows us to integrate these term-by-term. However, more important is usually the corresponding statement for differentiation. The following is a direct consequence of Theorem 46.2:

Corollary 46.3 *Let $\{a_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ be such that*

$$R := \Bigl[\limsup_{n\to\infty} |a_n|^{1/n}\Bigr]^{-1} > 0 \tag{46.19}$$

with $+\infty^{-1} := 0$ and $0^{-1} := +\infty$. Let $x_0 \in \mathbb{R}$. Then $f\colon (x_0-R, x_0+R) \to \mathbb{R}$ defined by $f(x) := \sum_{n=0}^{\infty} a_n(x-x_0)^n$ is arbitrary many times differentiable with the k-th derivative satisfying

$$f^{(k)}(x) = \sum_{n=k}^{\infty}\left[\prod_{i=0}^{k-1}(n-i)\right] a_n(x-x_0)^n \tag{46.20}$$

where the series converges absolutely on (x_0-R, x_0+R) and uniformly on any compact subinterval thereof.

Proof. Let $k \in \mathbb{N}$ obey $k \geqslant 1$. Following the proof of Lemma 45.10, for each $r \in (0,R)$ and $\epsilon \in (0, R-r)$ there exists $A \in (0,\infty)$ such that (45.35) holds. This means that

$$\forall x \in [x_0-r, x_0+r]\colon\ \Bigl|\frac{\mathrm{d}^k}{\mathrm{d}x^k} a_n(x-x_0)^n\Bigr| \leqslant A r^{-k} n^k \Bigl(\frac{r}{R-\epsilon}\Bigr)^n \tag{46.21}$$

The right-hand side is summable on n and so the series of k-th derivatives converges uniformly on $[x_0-r, x_0+r]$ by the Weierstrass M-test. The claim now follows by an inductive application of Theorem 46.2. □

We now use these to prove:

Lemma 46.4 (Exponential, sine and cosine) *The real-valued functions* exp, sin *and* cos *are well-defined for all $x \in \mathbb{R}$ by*

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{46.22}$$

$$\sin(x) := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{46.23}$$

$$\cos(x) := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{46.24}$$

where the series converge absolutely everywhere and uniformly on any compact subinterval of $\mathbb{R}$. *Moreover, these functions are arbitrary many times differentiable on $\mathbb{R}$ with the derivatives*

$$\exp'(x) = \exp(x) \ \wedge\ \sin'(x) = \cos(x) \ \wedge\ \cos'(x) = -\sin(x) \tag{46.25}$$

at each $x \in \mathbb{R}$.

Proof. We claim that the quantity R from (47.1) equals $+\infty$ for all three series. To see this, note that for $n \geqslant 3$ we have $n/3 \geqslant 1$ and $n - \lceil n/3 \rceil \geqslant n - n/3 - 1 \geqslant n/3$ and so

$$n! \geqslant \prod_{n/3 \leqslant i \leqslant n} i \geqslant (n/3)^{n/3} \tag{46.26}$$

and so $|n!|^{1/n} \geqslant (n/3)^{1/3}$. It follows that $\limsup_{n\to\infty} (1/n!)^{1/n} = 0$ thus proving that the radius of convergence of all three series is infinite. Corollary 47.1 allows us to differentiate the series term-ty-term which then readily shows (47.7). ☐

The exp function, to be called *exponential*, obeys the following:

Lemma 46.5 *The function* exp *defined above obeys*

$$\forall x, y \in \mathbb{R}\colon\ \ \exp(x+y) = \exp(x) \cdot \exp(y) \tag{46.27}$$

In particular, setting $\mathrm{e} := \sum_{n=0}^{\infty} \frac{1}{n!}$ *we have*

$$\forall x \in \mathbb{R}\colon\ \ \exp(x) = \mathrm{e}^x \tag{46.28}$$

where $\mathrm{e}^x := \inf\{\sqrt[q]{\mathrm{e}^p}\colon p \in \mathbb{Z} \wedge q \geqslant 1 \wedge x \leqslant p/q\}$.

We leave the proof of the lemma to a homework exercise while noting that the main idea is the following fact: If $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and obeys $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$ then either f vanishes identically everywhere, or there is $a > 0$ such that $f(x) = a^x$ for all $x \in \mathbb{R}$. The continuity assumption cannot be dropped; indeed, without that other solutions that are not of this form exist.

For the sine and cosine functions we in turn get:

Lemma 46.6 *The* sin *and* cos *functions defined above satisfy:*

(1) $\forall x \in \mathbb{R}\colon\ \sin(x)^2 + \cos(x)^2 = 1$ *and so* sin *and* cos *take values in* $[-1, 1]$,
(2) *the addition formulas hold:*

$$\forall x, y \in \mathbb{R}\colon\quad \begin{aligned} \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\ \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \end{aligned} \tag{46.29}$$

(3) *The number*

$$\pi := 2\inf\{t \geqslant 0\colon\ \cos(t) = 0\} \tag{46.30}$$

obeys $\pi \in (0,\infty)$ *and we have*

$$\forall x \in \mathbb{R}\colon\quad \sin(x) = -\cos(x+\pi/2) = -\sin(x-\pi) \tag{46.31}$$

and so, in particular,

$$\forall x \in \mathbb{R}\colon\quad \sin(x+2\pi) = \sin(x)\ \wedge\ \cos(x+2\pi) = \cos(x) \tag{46.32}$$

showing that sin *and* cos *are* 2π*-periodic.*

Note that the above defines two fundamental numbers in analysis; the Euler number e, named after early 18-th century Swiss mathematician and scientist Leonard Euler, and the Ludolphine number π, named after 16-th century German/Dutch mathematician Ludolph van Ceulen who computed π to 35 digits.

The definition $\mathrm{e} := \sum_{n=0}^{\infty}\frac{1}{n!}$ makes it easy to check that $\mathrm{e} \in (2,3)$. For π we use that sin has to be increasing and non-negative on $[0,\pi/2]$ and, being equal to negative of its second derivative, also concave. This, along with its taking values in $[-1,1]$ shows $2x/\pi \leqslant \sin(x) \leqslant 1$. Integrating over $[0,\pi/2]$ then shows

$$\frac{\pi}{4} = \frac{x^2}{\pi}\Big|_0^{\pi/2} \leqslant \int_0^{\pi/2}\sin(x)\mathrm{d}x \leqslant x\Big|_0^{\pi/2} = \frac{\pi}{2} \tag{46.33}$$

which in light of $1 = \cos(0)-\cos(\pi/2) = \int_0^{\pi/2}\sin(x)\mathrm{d}x$ gives $\pi \in [2,4]$. If we instead use the upper bound $\sin(x) \leqslant x$, we even get $\pi^2 \geqslant 8$, i.e., $\pi \geqslant \sqrt{8}$.

The Euler and Ludoplhine numbers are known to extremely high accuracy:

$$\begin{aligned} \mathrm{e} &= 2.71828182845904523536028747135 27\ldots \\ \pi &= 3.14159265358979323846264338327950288\ldots \end{aligned} \tag{46.34}$$

Both numbers are irrational which, for e, is actually easy to show:

Lemma 46.7 $\mathrm{e} \notin \mathbb{Q}$

Proof. If e were rational, we could write it as $\mathrm{e} = p/q$ for some natural $p, q \geqslant 2$. Then $q!\mathrm{e}$ is an integer and, since $n!$ divides $q!$ when $n \leqslant q$, also

$$\sum_{n=q+1}^{\infty}\frac{q!}{n!} = q! - \sum_{n=0}^{q}\frac{q!}{n!} \tag{46.35}$$

is an integer. But $q \geqslant 2$ implies $q!/n! \geqslant \frac{1}{q+1}2^{-(n-q-1)}$ for $n \geqslant q+1$ and so

$$0 < \sum_{n=q+1}^{\infty}\frac{q!}{n!} \leqslant \sum_{n=q+1}^{\infty}\frac{1}{q+1}2^{-(n-q-1)} = \frac{1}{q+1}\sum_{k=0}^{\infty}2^{-k} = \frac{2}{q+1} < 1 \tag{46.36}$$

a contradiction. So we must have $\mathrm{e} \notin \mathbb{Q}$ after all. □

A curious fact (discovered by L. Euler) is that π and e are closely related. Indeed, the suspiciously similar form of the power series (47.4–47.6) is resolved into

$$\forall x \in \mathbb{R}\colon\ \mathrm{e}^{\mathrm{i}x} = \cos(x) + \mathrm{i}\sin(x) \tag{46.37}$$

where i is the imaginary number such that $\mathrm{i}^2 = -1$. (This requires defining these functions for complex-valued arguments; the proof that the defining series converge is identical to the real case.) In particular, from $\sin(\pi) = 0$ and $\cos(\pi) = -1$ we have

$$\mathrm{e}^{\mathrm{i}\pi} = -1 \tag{46.38}$$

The latter is merely a restatement of the definition of π (the same formula holds with any odd multiple of π substituted for π), but the *Euler formula* (47.19) is deeper because serves as the basis of the polar representation of complex numbers.

We will return to power series representations of functions (not just transcendental ones) when we discuss analytic functions. This will allow us to give an infinite series representation for π as well albeit not so rapidly convergent as the one of e that the argument from Lemma 47.5, which incidentally applies to any number of the form $\sum_{n=0}^{\infty} \frac{a_n}{n!}$ with $\{a_n\}_{n\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}}$ of at most exponential growth, could be readily repeated.