45. Uniform convergence

Having discussed the salient aspects of differential and integral calculus, we will move to a new topic; namely, uniform convergence of functions. We then give applications to functions defined by uniformly convergent series.

45.1 Uniform convergence.

A ubiquitous but important problem in analysis is exchange of limits. As an example, we are interested to know under what conditions a two dimensional array $\{a_{m,n}\}_{m,n\in\mathbb{N}}$ of real numbers satisfies

$$\lim_{m\to\infty}\lim_{n\to\infty} a_{m,n} = \lim_{n\to\infty}\lim_{m\to\infty} a_{m,n} \tag{45.1}$$

assuming that the inner limits both exist. Writing $A := \{(k,\ell) \in \mathbb{N}\times\mathbb{N} \colon k \leqslant \ell\}$ for which $\lim_{m\to\infty} 1_A(m,n) = 0$ and $\lim_{n\to\infty} 1_A(m,n) = 1$ shows that (45.1) definitely does not hold automatically. The problem clearly stems from the fact that the larger the m the larger the n needs to be taken for $a_{m,n}$ to be close to its $n\to\infty$ limit. This can be circumvented by requiring the closeness of $a_{m,n}$ to its $n\to\infty$ limit uniformly in m:

Lemma 45.1 (Exchange of limits) *Let* $\{a_{m,n}\}_{m\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ *obey*

$$\forall m\in\mathbb{N}\colon\ b_m := \lim_{n\to\infty} a_{m,n} \text{ exists} \tag{45.2}$$

and

$$\forall n\in\mathbb{N}\colon\ c_n := \lim_{m\to\infty} a_{m,n} \text{ exists} \tag{45.3}$$

If, in addition to (45.2)*, we also have*

$$\lim_{n\to\infty}\sup_{m\in\mathbb{N}} |b_m - a_{m,n}| = 0 \tag{45.4}$$

then

$$\lim_{m\to\infty} b_m \text{ exists} \wedge \lim_{n\to\infty} c_n \text{ exists} \wedge \lim_{m\to\infty} b_m = \lim_{n\to\infty} c_n \tag{45.5}$$

All the limits are in $\mathbb{R}$ *with the supremum in* (45.4) *thus forced to be finite for large-enough* n*.*

We leave the easy proof of this lemma to a homework exercise. Note that (45.4) strengthens (45.2) to the desired uniformity (in m). Notwithstanding, our main interest in uniformity is the context of sequences of functions:

Definition 45.2 (Pointwise and uniform convergence) *Given sets* $A \subseteq B$*, a metric space* (X,ρ)*, a function* $f\colon B\to X$ *and a sequence of functions* $\{f_n\}_{n\in\mathbb{N}}$ *from* B *to* X *(with* $\mathrm{Dom}(f) = B$ *and* $\forall n\in\mathbb{N}\colon \mathrm{Dom}(f_n) = B$*), we say:*

- $f_n \to f$ pointwise on A *if*

$$\forall x\in A\colon\ \lim_{n\to\infty}\rho\big(f_n(x), f(x)\big) = 0 \tag{45.6}$$

- $f_n \to f$ uniformly on A *if*

$$\lim_{n\to\infty}\sup_{x\in A}\rho\big(f_n(x), f(x)\big) = 0 \tag{45.7}$$

If $A = B$ *then "on* A*" suffix is usually dropped.*

The reader will notice that uniform convergence is a metric-convergence with respect to the *supremum metric*

$$\rho_\infty(f,g) := \sup_{x\in A} \rho\bigl(f(x),g(x)\bigr) \tag{45.8}$$

with the only potential caveat being that the supremum needs to be finite for all pairs of functions of interest.

It is immediate that uniform convergence implies pointwise convergence:

Lemma 45.3 *For any $\{f_n\}_{n\in\mathbb{N}}$ and f functions $A \to X$,*

$$f_n \to f \text{ uniformly} \;\Rightarrow\; f_n \to f \text{ pointwise} \tag{45.9}$$

Proof. The uniform convergence implies that, given $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$\forall n \geqslant n_0\colon \sup_{x\in A} \rho\bigl(f_n(x), f(x)\bigr) < \epsilon \tag{45.10}$$

But then for any $z \in A$, we have

$$\forall n \geqslant n_0\colon \rho\bigl(f_n(z), f(z)\bigr) < \epsilon \tag{45.11}$$

thus showing that $\rho(f_n(z), f(z)) \to 0$ and proving pointwise convergence. □

We note that another way to see (45.9) is by writing the definitions (45.6–45.7) in logical primitives as

$$f_n \to f \text{ pointwise} \;:=\; \forall \epsilon > 0\, \forall x \in X\, \exists n_0 \in \mathbb{N}\, \forall n \geqslant n_0\colon\;\; \rho\bigl(f_n(x), f(x)\bigr) < \epsilon \tag{45.12}$$

while

$$f_n \to f \text{ uniformly} \;:=\; \forall \epsilon > 0\, \exists n_0 \in \mathbb{N}\, \forall x \in X\, \forall n \geqslant n_0\colon\;\; \rho\bigl(f_n(x), f(x)\bigr) < \epsilon \tag{45.13}$$

and so the "mere" difference is the swap of "$\forall x \in X$" and "$\exists n_0 \in \mathbb{N}$." We have actually seen this kind of a swap in the definition of uniform continuity (vs continuity).

45.2 Relation to continuity.

In order to give an example of these notions, consider the functions $\{f_n\}_{n\in\mathbb{N}}$ of the type $(0,\infty) \to \mathbb{R}$ defined by

$$f_n(x) := \frac{nx}{1+nx} \tag{45.14}$$

Then $f_n(x) \to 1$ for $x > 0$ yet $f_n(0) = 0$ for all $n \in \mathbb{N}$ and so $f_n \to 1_{(0,\infty)}$ pointwise. Note, however, that the convergence is not uniform because $\sup_{x>0} |f_n(x) - 1| = 1$ for all $n \in \mathbb{N}$, due to the Intermediate Value Theorem and the fact that all of these functions tend continuously to zero as $x \to 0^+$.

At this point it is interesting to note that, while each f_n in (45.14) is continuous, the limit function $1_{(0,\infty)}$ is not. As our next theorem shows, already this is enough to invalidate uniform convergence:

Theorem 45.4 (Continuity preserved by uniform convergence) *Given metric spaces X and Y, let $\{f_n\}_{n\in\mathbb{N}}$ and f be functions $X \to Y$ (with domain X) such that $f_n \to f$ uniformly. Then*

$$\forall x_0 \in X\colon\; \bigl(\forall n \in \mathbb{N}\colon f_n \text{ continuous at } x_0\bigr) \;\Rightarrow\; f \text{ continuous at } x_0 \tag{45.15}$$

In particular, the limit of a uniformly convergent sequence of continuous functions is continuous.

Proof. The proof uses a so called 3ϵ-argument. Let $\epsilon > 0$. The fact that $f_n \to f$ uniformly implies the existence of $n \in \mathbb{N}$ such that

$$\forall x \in X\colon\ \rho_Y\bigl(f_n(x), f(x)\bigr) < \epsilon \tag{45.16}$$

The fact that f_n is continuous at x_0 in turn gives a $\delta > 0$ such that

$$\forall x \in X\colon\ \rho_X(x, x_0) < \delta \ \Rightarrow\ \rho_Y\bigl(f_n(x), f_n(x_0)\bigr) < \epsilon \tag{45.17}$$

The triangle inequality now shows

$$\begin{aligned}\rho_Y\bigl(f(x), f(x_0)\bigr) \leqslant \rho_Y\bigl(f(x), f_n(x)\bigr)& \\ + \rho_Y\bigl(f(x_0), f_n(x_0)\bigr) + \rho_Y\bigl(f_n(x), f_n(x_0)\bigr)&\end{aligned} \tag{45.18}$$

For x such that $\rho_X(x, x_0) < \delta$, (45.16–45.17) show that each term on the right is $< \epsilon$ and so $\rho_Y(f(x), f(x_0)) < 3\epsilon$. As ϵ was arbitrary, we have shown that f continuous at x_0. □

The statement is not limited to continuity; in fact, a relatively minor variation on the proof of Theorem 45.4 gives:

Corollary 45.5 *Let X and Y be metric spaces, $\{f_n\}_{n\in\mathbb{N}}$ and f functions $X \to Y$ (with domain X) such that $f_n \to f$ uniformly. Let $x_0 \in X$ be such that*

$$\forall n \in \mathbb{N}\colon\ \lim_{x\to x_0} f_n(x) \text{ exists} \tag{45.19}$$

Assuming, in addition, that Y is complete, we then have

$$\lim_{x\to x_0} f(x) \text{ exists} \ \wedge\ \lim_{n\to\infty}\lim_{x\to x_0} f_n(x) \text{ exists} \tag{45.20}$$

and

$$\lim_{x\to x_0} f(x) = \lim_{n\to\infty}\lim_{x\to x_0} f_n(x) \tag{45.21}$$

We leave the proof as an exercise. The reader has likely found the statement very similar to that of Lemma 45.1. Indeed, it can be made to subsume it provided we "compactify" $\mathbb{N}$ by adding a point at infinity.

45.3 Application to uniformly convergent series.

In the proofs we often need to consider sequences of functions whose limit is not (yet) available. uniform convergence:

Definition 45.6 *Let A be a set and (X, ρ) a metric space. A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ from A to X is said to be* uniformly Cauchy *if*

$$\forall \epsilon > 0\, \exists n_0 \in \mathbb{N}\, \forall m, n \geqslant n_0 \forall x \in A\colon\ \rho\bigl(f_n(x), f_m(x)\bigr) < \epsilon \tag{45.22}$$

To see how this ties to uniform convergence, we prove:

Lemma 45.7 *Let $\{f_n\}_{n\in\mathbb{N}}$ and f be functions from a set A to a metric space (X, ρ). Then*

$$f_n \to f \text{ uniformly} \ \Rightarrow\ \{f_n\}_{n\in\mathbb{N}} \text{ uniformly Cauchy} \tag{45.23}$$

On the other hand, if (X, ρ) is complete, then

$$\{f_n\}_{n\in\mathbb{N}} \text{ uniformly Cauchy} \ \Rightarrow\ \exists f \in X^A\colon\ f_n \to f \text{ uniformly} \tag{45.24}$$

Proof. We will start with (45.23). Let $\epsilon > 0$. Then

$$\sup_{x\in X}\rho\big(f_n(x),f_m(x)\big) \leqslant \sup_{x\in X}\rho\big(f_n(x),f(x)\big)+\sup_{x\in X}\rho\big(f(x),f_m(x)\big) \tag{45.25}$$

Under the uniform convergence $f_n \to f$, both terms on the right are smaller than ϵ once m and n are sufficiently large. This is exactly what is required by (45.22).

Moving to (45.24), here we first observe that the assumption that $\{f_n\}_{n\in\mathbb{N}}$ is uniformly Cauchy implies that $\{f_n(x)\}_{n\in\mathbb{N}}$ is Cauchy for all $x \in X$. Thanks to the assumed completeness, the pointwise limit $f(x) := \lim_{n\to\infty} f_n(x)$ exists for each x. Passing to the limit $m \to \infty$ in (45.22) we get

$$\rho\big(f_n(x),f(x)\big) = \lim_{m\to\infty}\rho\big(f_n(x),f_m(x)\big) \tag{45.26}$$

Since the quantity on the right is less than ϵ once $n \geqslant n_0$, hence we get that $f_n \to f$ uniformly. □

The second part of the previous proof can be partially summarized as:

Corollary 45.8 *Let $\{f_n\}_{n\in\mathbb{N}}$ and f be functions from a set A to a metric space (X,ρ). Then*

$$f_n \to f \text{ pointwise} \wedge \{f_n\}_{n\in\mathbb{N}} \text{ uniformly Cauchy} \;\Rightarrow\; f_n \to f \text{ uniformly} \tag{45.27}$$

The above shows that, modulo extraction of a limit function (which is where the completeness is needed), uniform Cauchy property is necessary and sufficient for a pointwise limit being a uniform limit. Phrasing statements using the uniform Cauchy property allows us to not to commit to there being a limit function right from the start.

In order to demonstrate the connection between continuity and uniform convergence, we will study real-valued functions defined by infinite series of the form $\sum_{n=0}^{\infty} f_n$. Here is a key tool in this endeavor:

Lemma 45.9 (Weierstrass M-test) *Let X be a metric space and let $\{f_n\}_{n\to\infty}$ be real-valued functions on X such that, for a sequence $\{M_n\}_{n\in\mathbb{N}} \in [0,\infty)^{\mathbb{N}}$,*

$$\forall x \in X \forall n \in \mathbb{N}\colon\; \big|f_n(x)\big| \leqslant M_n \tag{45.28}$$

and

$$\sum_{n=0}^{\infty} M_n < \infty \tag{45.29}$$

Then there exists a function $f\colon X \to \mathbb{R}$ such that

$$\sum_{k=0}^{n} f_k \to f \;\text{ uniformly} \tag{45.30}$$

(We will denote f as $\sum_{n=0}^{\infty} f_n$.)

Proof. Let $n \leqslant m$ be naturals. Then for all $x \in X$,

$$\Big|\sum_{k=0}^{n} f_k(x) - \sum_{k=0}^{n} f_k(x)\Big| \leqslant \sum_{k=n+1}^{m} \big|f_k(x)\big| \leqslant \sum_{k=n+1}^{\infty} M_k \tag{45.31}$$

The convergence (45.24) ensures that the sum on the right is less than ϵ once n is sufficiently large. As the bound does not depend on x, we get that the sequence of partial sums $\{\sum_{k=0}^n f_k\}_{n\in\mathbb{N}}$ is uniformly Cauchy. With the sequence is $\mathbb{R}$ valued, and $\mathbb{R}$ complete, the claim follows from Lemma 45.7. □

In order to give a concrete example, we consider functions given as power series:

Lemma 45.10 *Let $\{a_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ be such that*

$$R := \Bigl[\limsup_{n\to\infty} |a_n|^{1/n}\Bigr]^{-1} > 0 \tag{45.32}$$

with the convention $+\infty^{-1} := 0$ and $0^{-1} := +\infty$. Let $x_0 \in \mathbb{R}$. Then for all $x \in (x_0 - R, x_0 + R)$,

$$f(x) := \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{45.33}$$

converges absolutely and defines a continuous function $f\colon (x_0 - R, x_0 + R) \to \mathbb{R}$.

Proof. Let $r \in (0, R)$ and let $\epsilon \in (0, R - r)$. As $(R-\epsilon)^{-1} > R^{-1}$, the definition of R ensures that there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geqslant n_0\colon\ |a_n| \leqslant \frac{1}{(R-\epsilon)^n} \tag{45.34}$$

Denoting $A := \max(\{|a_k|(R-\epsilon)^k\colon k = 0, \dots, n_0\} \cup \{1\})$, we then have

$$\forall n \in \mathbb{N}\colon\ |a_n| \leqslant \frac{A}{(R-\epsilon)^n} \tag{45.35}$$

Denoting $f_n(x) := a_n(x - x_0)^n$, we then have

$$M_n := \sup_{x\in[x_0-r,x_0+r]} \bigl|f_n(x)\bigr| \leqslant |a_n| r^n \leqslant A\Bigl(\frac{r}{R-\epsilon}\Bigr)^n \tag{45.36}$$

Since $r < R - \epsilon$, this is summable on n and so the Weierstrass M-test shows that the the series defining f converges pointwise absolutely as well as uniformly on $[x_0 - r, x_0 + r]$. By Theorem 45.4, the limit function is continuous on $[x_0 - r, x_0 + r]$. Since this holds for all $r \in (0, R)$, we get the claim. □

We finish by noting that, while the above criterion for uniform convergence also implies absolute convergence, these are distinct notions. For instance, the series

$$f(x) := \sum_{n=1}^{\infty} \frac{(-1)^n}{n} 1_{[0,n]}(x) \tag{45.37}$$

converges uniformly on $\mathbb{R}$ yet not pointwise absolutely at any point.