

44. CONDITIONS FOR STIELTJES INTEGRABILITY

The aim of this section is to give reasonable sufficient conditions for Stieltjes integrability. The material has not really been lectured on and is included for the readers interested to delve deeper into the subject.

44.1 Sufficient conditions for Stieltjes integrability.

In our discussion of the Stieltjes integral, we have so far given only one necessary condition for integrability (namely, (43.14) in Lemma 43.6). A simple necessary and sufficient condition is:

Lemma 44.1 (Cauchy criterion for Stieltjes integral) *Let $f, g: [a, b] \rightarrow \mathbb{R}$. Then $f \in \text{RS}(g, [a, b])$ (and equivalently $g \in \text{RS}(f, [a, b])$) if and only if for each $\epsilon > 0$ there is a $\delta > 0$ such that for all marked partitions Π, Π' of $[a, b]$,*

$$\max\{\|\Pi\|, \|\Pi'\|\} < \delta \Rightarrow |S(f, dg, \Pi) - S(f, dg, \Pi')| < \epsilon \quad (44.1)$$

Proof. This is a direct consequence of the definition of Stieltjes integrability. \square

A somewhat deeper sufficient criterion concerns Stieltjes integrals with respect to functions of bounded variation:

Lemma 44.2 *Let $f, g: [a, b] \rightarrow \mathbb{R}$ be such that f is bounded and g is of bounded variation, i.e., $V(g, [a, b]) < \infty$. If for each $\epsilon > 0$ there is a partition $\Pi = \{t_i\}_{i=0}^n$ of $[a, b]$ such that*

$$\sum_{i=1}^n \text{osc}(f, [t_{i-1}, t_i]) V(g, [t_{i-1}, t_i]) < \epsilon \quad (44.2)$$

then $f \in \text{RS}(g, [a, b])$.

Proof idea. Using that, by Jordan's decomposition, g of bounded variation can be written as a difference of non-decreasing functions $h_1(x) := V(g, [a, x])$ and $h_2(x) := V(g, [a, x]) - g(x)$, the fact that

$$h_1(t_i) - h_1(t_{i-1}) = V(g, [t_{i-1}, t_i]) \quad (44.3)$$

and

$$h_2(t_i) - h_2(t_{i-1}) = V(g, [t_{i-1}, t_i]) - g(t_i) + g(t_{i-1}) \leq V(g, [t_{i-1}, t_i]) \quad (44.4)$$

the condition in the statement implies

$$\sum_{i=1}^n \text{osc}(f, [t_{i-1}, t_i]) (h(t_i) - h(t_{i-1})) < 2\epsilon \quad (44.5)$$

for $h = h_1, h_2$. We can now proceed similarly as for the Riemann integral with some additional care paid to discontinuities of f and h_1 and h_2 . (The assumption rules out that these occur at the same point.) \square

proof of this lemma is similar to that un derly we conclude:

Corollary 44.3 Let $f, g: [a, b] \rightarrow \mathbb{R}$ be such that f is continuous on $[a, b]$ and g is of bounded variation on $[a, b]$. Then $f \in \text{RS}(g, [a, b])$ as well as $g \in \text{RS}(f, [a, b])$ and

$$\left| \int_a^b f dg \right| \leq \left(\sup_{x \in [a, b]} |f(x)| \right) V(g, [a, b]) \quad (44.6)$$

Proof. The integrability of f with respect to g is proved using the bound Lemma 44.2 combined with the fact that, by uniform continuity of f , for each $\epsilon > 0$ there is $\delta > 0$ such that $\text{osc}(f, [s, t]) < \epsilon/(b-a)$ whenever $t-s < \delta$. The integrability of g with respect to f then follows from Lemma 43.9. The bound on the integral inherited from the corresponding bound on $S(f, dg, \Pi)$. \square

Note that the additivity of the integral then implies that, for $f, \tilde{f}: [a, b] \rightarrow \mathbb{R}$ continuous and $g, \tilde{g}: [a, b] \rightarrow \mathbb{R}$ bounded variation,

$$\left| \int_a^b f dg - \int_a^b \tilde{f} d\tilde{g} \right| \leq (\sup |f - \tilde{f}|) V(g, [a, b]) + (\sup |f|) V(g - \tilde{g}, [a, b]) \quad (44.7)$$

This is a statement of continuity of $f \mapsto \int_a^b f dg$ in the supremum norm $f \mapsto \sup |f|$ and the continuity of $g \mapsto \int_a^b f dg$ in the total variation norm $g \mapsto V(g, [a, b])$. Note that, in light of Lemma 43.9, the same continuity applies to the integral $\int_a^b g df$.

Lemma 44.4 Let $f, g: [a, b] \rightarrow \mathbb{R}$ be functions and assume that f is bounded and g is of bounded variation; i.e., $V(g, [a, b]) < \infty$. Let $v_g: [a, b] \rightarrow \mathbb{R}$ be defined by $v_g(t) := V(g, [a, t])$. Then

$$f \in \text{RS}(g, [a, b]) \Leftrightarrow f \in \text{RS}(v_g, [a, b]) \quad (44.8)$$

and, if both TRUE, then also $|f| \in \text{RS}(v_g, [a, b])$ and

$$\left| \int_a^b f dg \right| \leq \int_a^b |f| dv_g \quad (44.9)$$

We leave the proof of this lemma, with g assumed continuous, to homework. Discontinuities of g are handled by a separate argument. The generalized Stieltjes integrability fares better in this context.

44.2 Young integral.

The assumption that f is continuous and g is bounded variation, or *vice versa*, is the one most commonly made in the literature on the Stieltjes integral. However, this is not the end of the story; indeed, one can trade regularity of g against regularity of f . This was pushed by L.C. Young in the 1930s and reappeared quite usefully in stochastic analysis over the last two decades. The starting point is an inequality that uses the notion of p -variation of $f: [a, b] \rightarrow \mathbb{R}$ defined, for $p > 0$, by

$$V^p(f, [a, b]) := \sup_{n \geq 1} \sup_{\Pi = \{t_i\}_{i=0}^n} \sum_{i=1}^n |f(t_i) - f(t_{i-1})|^p \quad (44.10)$$

We then have:

Lemma 44.5 (Love-Young inequality) *Let $a < b$ be reals and $f, g: [a, b] \rightarrow \mathbb{R}$ functions such that $V^p(f, [a, b]) < \infty$ and $V^q(g, [a, b]) < \infty$. Then for all $p, q > 0$, all natural $n \geq 1$, all marked partition $\Pi = (\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n)$ of $[a, b]$ into n intervals and all $t \in [a, b]$,*

$$\begin{aligned} & \left| S(f, dg, \Pi) - f(t)[g(b) - g(a)] \right| \\ & \leq \left(1 + \sum_{k=1}^{n-1} \frac{1}{k^{1/p+1/q}} \right) V^p(f, [a, b])^{1/p} V^q(g, [a, b])^{1/q}, \end{aligned} \tag{44.11}$$

where the sum over k is treated as zero when $n = 1$.

Proof. The proof (drawn vaguely from L.C. Young’s paper “An inequality of the Hölder type, connected with Stieltjes integration” in *Acta Mathematica* in 1938) hinges on the following observation: Given a natural $n \geq 1$ and reals $a_1, \dots, a_n, b_1, \dots, b_n \in [0, \infty)$, let $k = 1, \dots, n$ be such that $a_k b_k = \min_{i=1, \dots, n} a_i b_i$. Then the multivariate AMGM inequality

$$\forall x_1, \dots, x_n \geq 0: \left(\prod_{i=1}^n x_i \right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^n x_i \tag{44.12}$$

gives, for each $p, q > 0$, that

$$a_k b_k \leq \left(\prod_{i=1}^n a_i^p \right)^{\frac{1}{pn}} \left(\prod_{i=1}^n b_i^q \right)^{\frac{1}{qn}} \leq \frac{1}{n^{1/p+1/q}} \left(\sum_{i=1}^n a_i^p \right)^{1/p} \left(\sum_{i=1}^n b_i^q \right)^{1/q} \tag{44.13}$$

As we will see, this opens up the possibility to prove the claim by induction.

Fix $t \in [a, b]$ and reals $p, q > 0$. For the base case $n = 1$ of partition Π consisting of just one interval $[a, b]$ and a partition point t' , we have

$$S(f, dg, \Pi) - f(t)[g(a) - g(b)] = [f(t') - f(t)][g(b) - g(a)] \tag{44.14}$$

Assuming, without simplicity of notation, that $t' > t$, then

$$\begin{aligned} |f(t') - f(t)| &= \left(|f(t') - f(t)|^p \right)^{1/p} \\ &\leq \left(|f(t) - f(a)|^p + |f(t') - f(t)|^p + |f(b) - f(t')|^p \right)^{1/p} \\ &\leq V^p(f, [a, b])^{1/p} \end{aligned} \tag{44.15}$$

and using that, trivially, $|g(b) - g(t)| \leq V^q(g, [a, b])^{1/q}$ gives

$$\left| S(f, dg, \Pi) - f(t)[g(a) - g(b)] \right| \leq V^p(f, [a, b])^{1/p} V^q(g, [a, b])^{1/q} \tag{44.16}$$

thus proving the claim for $n = 1$.

Next suppose that the claim holds for a natural n and let $\Pi = (\{t_i\}_{i=0}^{n+1}, \{t_i^*\}_{i=1}^{n+1})$ be a partition of $[a, b]$ into $n + 1$ intervals. Let $k = 1, \dots, n$ be the smallest index such that

$$[f(t_{k+1}^*) - f(t_k^*)][g(t_k) - g(t_{k-1})] = \min_{i=1, \dots, n} [f(t_{i+1}^*) - f(t_i^*)][g(t_i) - g(t_{i-1})] \tag{44.17}$$

Now let Π' be a partition obtained by removing partition point t_i and marked point t_i^* from Π . The intervals $[t_{i-1}, t_i]$ and $[t_i, t_{i+1}]$ in Π are thus united into $[t_{i-1}, t_{i+1}]$ in Π' and

the latter interval now receives marked point t_{i+1}^* . As all other intervals and marked points remain the same, this gives

$$\begin{aligned} S(f, dg, \Pi') - S(f, dg, \Pi) &= f(t_{k+1}^*)[g(t_{k+1}) - g(t_{k-1})] \\ &\quad - f(t_{k+1}^*)[g(t_{k+1}) - g(t_k)] - f(t_k^*)[g(t_k) - g(t_{k-1})] \quad (44.18) \\ &= [f(t_{k+1}^*) - f(t_k^*)][g(t_k) - g(t_{k-1})] \end{aligned}$$

The inequality (44.10) enabled by (44.14) then gives

$$\begin{aligned} &|S(f, dg, \Pi') - S(f, dg, \Pi)| \\ &\leq \frac{1}{n^{1/p+1/q}} \left(\sum_{j=1}^n |f(t_{j+1}^*) - f(t_j^*)|^p \right)^{1/p} \left(\sum_{j=1}^n |g(t_j) - g(t_{j-1})|^q \right)^{1/q} \quad (44.19) \\ &\leq \frac{1}{n^{1/p+1/q}} V^p(f, [a, b])^{1/p} V^q(g, [a, b])^{1/q} \end{aligned}$$

Using that

$$\begin{aligned} &|S(f, dg, \Pi) - f(t)[g(b) - g(a)]| \\ &\leq |S(f, dg, \Pi') - S(f, dg, \Pi)| + |S(f, dg, \Pi') - f(t)[g(b) - g(a)]| \quad (44.20) \end{aligned}$$

the claim for Π follows by combining (44.16) with the claim for Π' , which is TRUE thanks to the induction assumption and the fact that Π' partitions $[a, b]$ into only n intervals. \square

We now put the above inequality to a good use in:

Theorem 44.6 (L.C. Young) *Let $a < b$ be reals and $f, g: [a, b] \rightarrow \mathbb{R}$ functions such that f is α -Hölder and g is β -Hölder for some $\alpha, \beta > 0$ with $\alpha + \beta > 1$. Then $f \in \text{RS}(g, [a, b])$ and $g \in \text{RS}(f, [a, b])$.*

Proof. Let $p > 1/\alpha$ and $q > 1/\beta$ be such that $\frac{1}{p} + \frac{1}{q} > 1$. (This is possible thanks to the assumption $\alpha + \beta > 1$.) The assumption that f is α -Hölder means that there exists $c > 0$ such that $\forall x, y \in [a, b]: |f(x) - f(y)| \leq c|x - y|^\alpha$. Pick any interval $[s, t] \subseteq [a, b]$ and let $\Pi = \{t_i\}_{i=0}^n$ be a partition of $[s, t]$. Then

$$\begin{aligned} \sum_{i=1}^n |f(t_i) - f(t_{i-1})|^p &\leq c^p \sum_{i=1}^n |t_i - t_{i-1}|^{\alpha p} \\ &\leq c^p (t - s)^{\alpha p - 1} \sum_{i=1}^n |t_i - t_{i-1}| = c^p (t - s)^{\alpha p} \quad (44.21) \end{aligned}$$

and so

$$V^p(f, [s, t])^{1/p} \leq c|t - s|^\alpha \quad (44.22)$$

Similarly we get

$$V^q(g, [s, t])^{1/q} \leq \tilde{c}|t - s|^\beta \quad (44.23)$$

where \tilde{c} is the constant such that $\forall x, y \in [a, b]: |g(x) - g(y)| \leq \tilde{c}|x - y|^\beta$.

Let $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ be the Riemann zeta function where we note the series converges as soon as $s > 1$. Given $\epsilon > 0$, let $\delta > 0$ be such that

$$c\tilde{c}(b-a)[1 + \zeta(1/p + 1/q)]\delta^{\alpha+\beta-1} < \epsilon \tag{44.24}$$

and let $\Pi = (\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n)$ and $\Pi' = (\{t'_i\}_{i=0}^m, \{t_i^{**}\}_{i=1}^m)$ be partitions of $[a, b]$ satisfying $\|\Pi\|, \|\Pi'\| < \delta$. Assuming first that Π' obeys $\Pi \subseteq \Pi'$, for each $i = 1, \dots, m$, let Π'_i be the partition of $[t_{i-1}, t_i]$ induced by Π' . Lemma 44.5 now gives

$$\begin{aligned} & \left| S(f, dg, \Pi'_i) - f(t_i^*)[g(t_i) - g(t_{i-1})] \right| \\ & \leq [1 + \zeta(1/p + 1/q)] V^p(f, [t_{i-1}, t_i])^{1/p} V^q(g, [t_{i-1}, t_i])^{1/q} \\ & \leq c\tilde{c}[1 + \zeta(1/p + 1/q)](t_i - t_{i-1})^{\alpha+\beta} < \frac{\epsilon}{b-a}(t_i - t_{i-1}) \end{aligned} \tag{44.25}$$

where we also invoked (44.19–44.20) and used $t_i - t_{i-1} < \delta$ along with (44.21). Hereby we get

$$\begin{aligned} \left| S(f, dg, \Pi') - S(f, dg, \Pi) \right| & \leq \sum_{i=1}^n \left| S(f, dg, \Pi'_i) - f(t_i^*)[g(t_i) - g(t_{i-1})] \right| \\ & \leq \sum_{i=1}^n \frac{\epsilon}{b-a}(t_i - t_{i-1}) = \epsilon \end{aligned} \tag{44.26}$$

When Π' is not a refinement of Π , then by going to their common refinement we bound the difference instead by 2ϵ . As $\epsilon > 0$ was arbitrary, the Cauchy criterion (cf Lemma 44.1) now implies $f \in \text{RS}(g, [a, b])$ and, by Lemma 44.1, also $g \in \text{RS}(f, [a, b])$. \square

A slightly more sophisticated argument shows that the finiteness of the p -variation of f and q -variation of g for some $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} > 1$ are sufficient for integrability of f with respect to g and *vice versa*. Young also showed that assuming this for $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$ is not enough. The Stieltjes integral derived under these conditions is sometimes referred to as the *Young integral* even though what Young's work does is to provide a useful sufficient condition for Stieltjes integrability rather than defining a new integration theory in its own right.